Lectures on Quantum Algorithms
Quanterall Academy and
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Mathematical Basics

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https://en.wikipedia.org/wiki/Analytic_function_of_a_matrix

https://archive.siam.org/books/ot104/OT104HighamChapter1.pdf

Complex numbers and functions

$$z = a + ib, \quad z^* = a - ib, \quad a = \mathrm{Re}\, z, \quad b = \mathrm{Im}\, z, \quad i^2 = -1,$$

$$z' = a' + ib', \qquad zz' = aa' - bb' + i(ab' + a'b), \quad z^2 = a^2 - b^2 + 2iab.$$

Polar representation of complex numbers:

$$z = |z|e^{i\theta} = \sqrt{a^2+b^2}(\cos(\theta){+}i\sin(\theta)), \quad z^* = |z|e^{-i\theta} = \sqrt{a^2+b^2}(\cos(\theta){-}i\sin(\theta)),$$

$$\theta = \arg(z), \qquad z' = |z'|e^{i\theta'} = \sqrt{a'^2+b'^2}(\cos(\theta') + i\sin(\theta'))$$

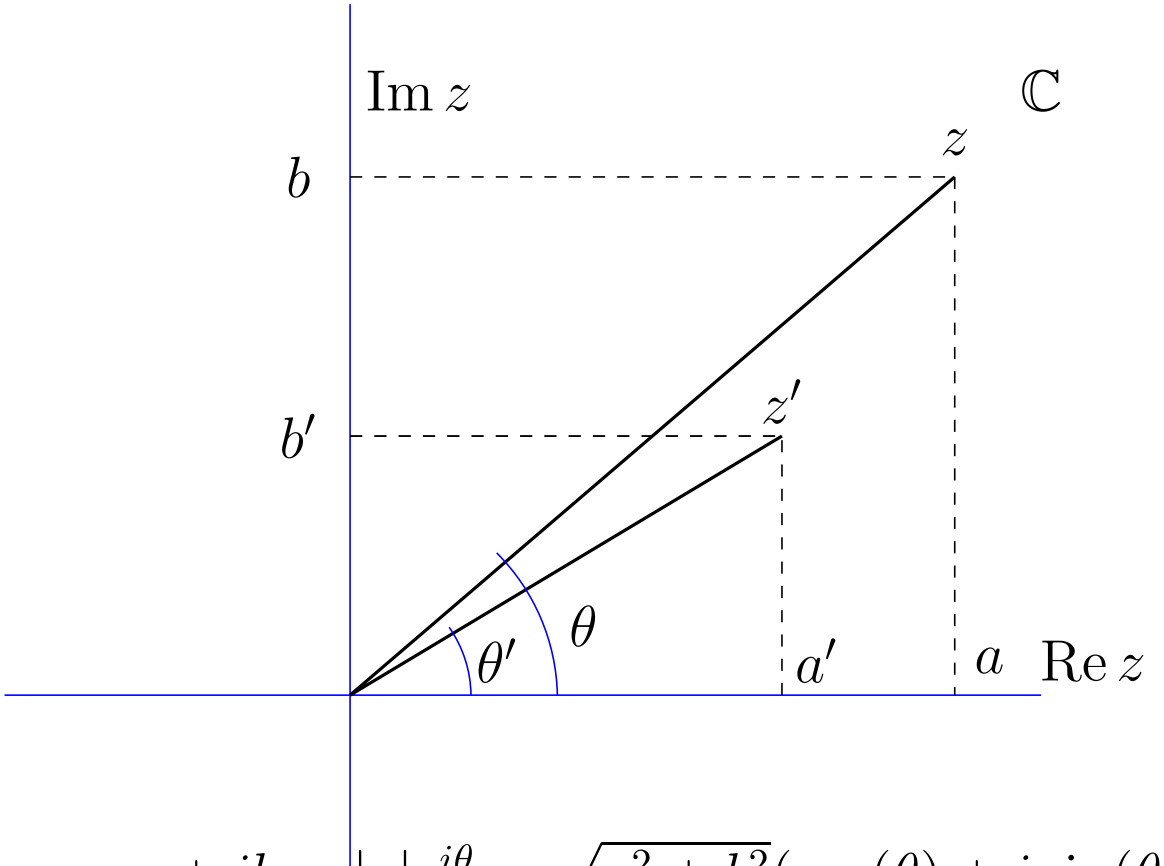
$$zz' = |z|\,|z'|e^{i(\theta+\theta')} = |z|\,|z'|(cos(\theta+\theta') + i\sin(\theta+\theta')).$$

Euler formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Euler identity:

$$e^{i\pi} + 1 = 0.$$



$$z = a + ib = |z|e^{i\theta} = \sqrt{a^2 + b^2}(\cos(\theta) + i\sin(\theta)),$$

$$z' = a' + ib' = |z'|e^{i\theta'} = \sqrt{a'^2 + b'^2}(\cos(\theta') + i\sin(\theta'))$$

$$zz' = |z|\,|z'|e^{i(\theta+\theta')}, \qquad e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

The field of complex numbers is closed! Namely: Any polynomial equation with complex coefficients a_k:

$$P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0,$$

has n complex roots $z_j = z_{j,0} + i z_{j,1}$, i.e.

$$P_n(z_j) = 0, \qquad j = 1, 2, \ldots, n.$$

The field of real numbers is not closed. For example, the equation:

$$x^2 + 1 = 0,$$

has NO real roots! Otherwise: $x_{1,2} = \pm i$.

Functions of complex variables.

Polynomials $P_n(z)$ can always be calculated using $z = a + ib$. Polynomials $P_n(1/z)$ can always be calculated using

$$\frac{1}{z} = \frac{z^*}{|z|^2} = \frac{a - ib}{a^2 + b^2} = \frac{e^{-i\theta}}{|z|}.$$

But there are other functions, e.g.

$$\sqrt{z}, \qquad \sqrt[n]{z} = z^{1/n}, \qquad \ln z, \quad \ldots$$

Here it is convenient to use the polar representation of z:

$$\sqrt{z} = \sqrt{|z|}e^{i\theta/2} = \sqrt{z}(\cos(\theta/2) + i\sin(\theta/2)),$$

$$\sqrt[n]{z} = \sqrt[n]{|z|}e^{i\theta/n} = \sqrt[n]{z}(\cos(\theta/n) + i\sin(\theta/n)),$$

$$\ln(z) = \ln|z| + i\arg(z) = \ln|z| + i\theta.$$

From the right-angle triangles we find:

$$|z| = \sqrt{a^2 + b^2},$$

$$\frac{b}{a} = \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta), \quad \theta = \arctan\frac{b}{a} = \arctan\frac{\operatorname{Im} z}{\operatorname{Re} z}.$$

Examples: Roots of unity:

$$\omega_3 = \sqrt[3]{1} = e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3) = \frac{-1+i\sqrt{3}}{2},$$

$$\omega_8 = \sqrt[4]{1} = e^{2\pi i/8} = \cos(2\pi/8) + i\sin(2\pi/8) = \frac{\sqrt{2}+i\sqrt{2}}{2},$$

$$\omega_5 = \sqrt[5]{1} = e^{2\pi i/5} = \cos(2\pi/5)+i\sin(2\pi/5) = \frac{1}{4}\left(\sqrt{5}-1+i\sqrt{10+2\sqrt{5}}\right),$$

$$\omega_6 = \sqrt[6]{1} = e^{2\pi i/6} = \cos(\pi/6) + i\sin(\pi/6) = \frac{1+i\sqrt{3}}{2},$$

$$\omega_{10} = \sqrt[10]{1} = e^{2\pi i/10} = \cos(2\pi/10)+i\sin(2\pi/10) = \frac{1}{4}\left(\sqrt{5}+1+i\sqrt{10-2\sqrt{5}}\right),$$

Check that:

$$\omega_3^3 = 1, \qquad \omega_8^2 = i, \qquad \omega_5\omega_5^* = 1, \qquad \omega_6^2 = \omega_3, \qquad \omega_6^3 = -1, \qquad \omega_{10}^2 = \omega_5$$

Basic properties of vectors and matrices

Simplest nontrivial: two-component vectors:

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = a\vec{e}_1 + b\vec{e}_2, \qquad \vec{x}' = \begin{pmatrix} a' \\ b' \end{pmatrix} = a'\vec{e}_1 + b'\vec{e}_2, \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Transposed vectors:

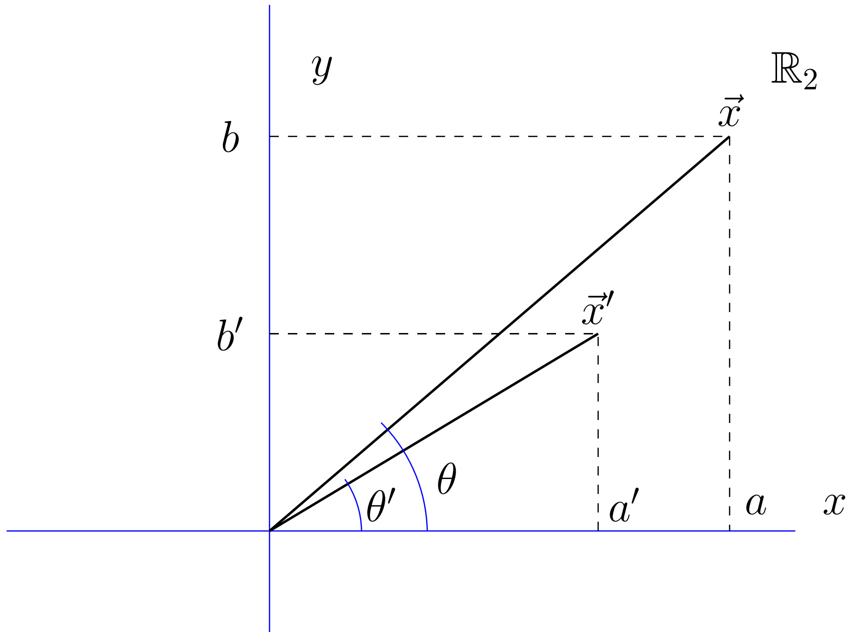
$$\vec{x}^T = (a, b) = a\vec{e}_1^T + b\vec{e}_2^T, \qquad \vec{e}_1^T = (1, 0), \qquad \vec{e}_2^T = (0, 1).$$

The Euclidean space $\mathbb{R}_2$ is a metric space:

$$|\vec{x}|^2 = (\vec{x}.\vec{x}) = \vec{x}^T\vec{x} = a^2 + b^2.$$

Polar coordinates:

$$\vec{x} = |\vec{x}| \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \qquad \vec{x}' = |\vec{x}'| \begin{pmatrix} \cos(\theta') \\ \sin(\theta') \end{pmatrix}, \quad |\vec{x}| = \sqrt{a^2 + b^2}, \quad |\vec{x}'| = \sqrt{a'^2 + b'^2},$$



$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \qquad (\vec{x}, \vec{x}) = |\vec{x}|^2 = a^2 + b^2,$$

$$\vec{x}' = \begin{pmatrix} a' \\ b' \end{pmatrix} = \sqrt{a'^2 + b'^2} \begin{pmatrix} \cos(\theta') \\ \sin(\theta') \end{pmatrix}, \qquad (\vec{x}', \vec{x}') = |\vec{x}'|^2 = a'^2 + b'^2,$$

$$(\vec{x}, \vec{x}') = aa' + bb' = |\vec{x}|\,|\vec{x}'| \cos(\theta - \theta'),$$

It allows scalar product:

$$\vec{x} = |\vec{x}| \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \qquad \vec{x}' = |\vec{x}'| \begin{pmatrix} \cos(\theta') \\ \sin(\theta') \end{pmatrix}, \quad |\vec{x}'| = \sqrt{a'^2 + b'^2}.$$

$$(\vec{x}.\vec{x}') = \vec{x}^T \vec{x}' = (a, b) \begin{pmatrix} a' \\ b' \end{pmatrix} = aa' + bb' = |\vec{x}|\,|\vec{x}'| \cos(\theta - \theta').$$

2×2 matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}, \qquad cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}.$$

Matrix multiplication:

$$AB = C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad c_{jk} = \sum_{m=1}^{2} a_{jm} b_{mk}.$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}, \qquad c_{12} = a_{11}b_{12} + a_{12}b_{22}, \ldots$$

Operations on matrices: complex conjugation, transposition, hermitian conjugation:

$$A^* = \begin{pmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{pmatrix}, \qquad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}, \qquad A^\dagger = (A^*)^T = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}.$$

Trace and determinant:

$$\mathrm{tr}\,(A) = \sum_{m=1}^{2} a_{mm}, \qquad \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2.$$

Properties:

$$\sigma_k^2 = \sigma_0 = \mathbb{1}_2, \quad \sigma_k \sigma_j = i\epsilon_{kjm}\sigma_m, \qquad \{k, j, m\} = \{1, 2, 3\}$$

where ϵ_{jkm} is totally antisymmetric tensor, i.e.

$$\epsilon_{123} = 1, \qquad \epsilon_{kjm} = -\epsilon_{jkm} = \epsilon_{mkj}.$$

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_2\sigma_1 = -i\sigma_3, \quad \sigma_3\sigma_2 = -i\sigma_1.$$

A special 2×2 matrix:

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \cos(\theta)\sigma_0 - i\sin(\theta)\sigma_2.$$

$$R_\theta \vec{x}' = |x'| \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta') \\ \sin(\theta') \end{pmatrix} = |x'| \begin{pmatrix} \cos(\theta + \theta') \\ \sin(\theta + \theta') \end{pmatrix}$$

i.e.R_θ **rotates every vector in $\mathbb{R}_2$ by angle θ.**

Pauli matrices and vectors. Consider:

$$X = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 - x_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix},$$

$$X' = x'_1\sigma_1 + x'_2\sigma_2 + x'_3\sigma_3 = \begin{pmatrix} x'_3 & x'_1 - x'_2 \\ x'_1 + ix'_2 & -x'_3 \end{pmatrix},$$

Prove that:

$$R_\theta R_{\theta'} = R_{\theta+\theta'}, \qquad R_\theta R_{-\theta} = \mathbb{1}_2, \qquad R_{-\theta} = R_\theta^T.$$

$$X^2 = (\vec{x}, \vec{x})\mathbb{1}_2, \qquad \mathrm{tr}\,(XX') = (\vec{x}, \vec{x}').$$

$n \times n$ matrices: general properties and definitions

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix},$$

$$A + B = S, \qquad s_{km} = a_{km} + b_{km}, \qquad (cA)_{km} = ca_{km},$$

$$AB = C, \qquad c_{km} = \sum_{p=1}^{n} a_{kp} b_{pm}, \qquad c_{23} = a_{21}b_{13}+a_{22}b_{23}+a24b_{43}+\dots+a_{2n}b_{n3}.$$

Operations: transpositions, complex conjugation, hermitian conjugation:

$$A = (a_{km})_{k,m=1}^{n}, \qquad A^T = (a_{mk})_{k,m=1}^{n},$$

$$A^* = (a_{km}^*)_{k,m=1}^{n}, \qquad A^\dagger = (A^T)^* = (a_{mk}^*)_{k,m=1}^{n},$$

$$\mathrm{tr}\,A = \sum_{m=1}^{n} a_{mm}, \qquad \det A = \sum_{k_1,k_2,\ldots,k_n}^{n} \epsilon_{k_1,k_2,\ldots,k_n} a_{1k_1} a_{2,k_2} \ldots a_{nk_n},$$

where $\epsilon_{k_1,k_2,\ldots,k_n}$ is n component antisymmetric tensor, i.e.:

$$\epsilon_{1,2,\ldots,n} = 1, \qquad \epsilon_{k_1,k_2,\ldots,k_p,k_{p+1},\ldots,k_n} = -\epsilon_{k_1,k_2,\ldots,k_{p+1},k_p,\ldots,k_n}.$$

Special kinds of matrices:

$$\text{unit matrix} \quad \mathbb{1}_n = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}, \qquad \text{traceless} \quad \mathrm{tr}\,A = 0,$$

$$\text{invertible} \quad \det A \neq 0, \qquad \text{diagonal} \quad \begin{pmatrix} \alpha_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \alpha_n \end{pmatrix},$$

If $\det A \neq 0$, then it is invertible, i.e. there exist unique matrix A^{-1} such that $A^{-1}A = AA^{-1} = \mathbb{1}_n$.

The matrix A is diagonalizable if there exist invertible matrix u such that

$$u^{-1}Au = \mathrm{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

The numbers α_n are called eigenvalues of A. In fact:

$$\mathrm{tr}\ A = \sum_{m=1}^{n} \alpha_m, \qquad \det A = \prod_{m=1}^{n} \alpha_m.$$

Characteristic polynomial:

$$P_A(z) = \det(A - z\mathbb{1}_n) = \prod_{m=1}^{n} (z - \alpha_m).$$

Groups of matrices: $\mathcal{G}$ is a group of $n \times n$ matrices if:
i) if X and Y are elements of $\mathcal{G}$ then $XY \in \mathcal{G}$;
ii) $\mathbb{1}_n \in \mathcal{G}$ is such that $\mathbb{1}_n X = X\mathbb{1}_n = X$
iii) if $X \in \mathcal{G}$ then $X^{-1} \in \mathcal{G}$.

Examples of groups:

$$GL(n)\colon X \in GL(n) \quad \text{iff} \quad \det X \neq 0;$$

$$SL(n)\colon X \in SL(n) \quad \text{iff} \quad \det X = 1;$$

$$SO(n)\colon X \in SO(n) \quad \text{iff} \quad XX^T = \mathbb{1}_n;$$

$$SU(n)\colon X \in SU(n) \quad \text{iff} \quad XX^\dagger = \mathbb{1}_n;$$

Functions of matrices

Polynomial functions: For any polynomial $P_N(z) = \sum_{m=1}^{N} p_m z^m$

$$P_N(A) = \sum_{m=1}^{N} p_m A^m.$$

Functions of diagonal matrices:

$$D = \operatorname{diag}(d_1, d_2, \ldots, d_n), \qquad P_N(D) = \operatorname{diag}(P_N(d_1), P_N(d_2), \ldots, P_N(d_n)),$$

$$f(D) = \text{diag}\,(f(d_1), f(d_2), \ldots, f(d_n)), \qquad f(z) \in \{z^{1/p}, \exp(z), \ln(z), \frac{z - s_1}{z - s_2}, \ldots, \}$$

Functions of diagonalizable matrices:

- Given diagonalizable matrix A we first diagonalize it:

$$A = u^{-1}\tilde{A}u, \qquad \tilde{A} = \text{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_n),$$

- evaluate $f(z)$ on $\tilde{A}$:

$$f(\tilde{A}) = \text{diag}\,(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n)).$$

  Note that the eigenvalues α_k typically are complex-valued. So, evaluating functions like $f(z) = z^{1/p}$, $f(z) = \exp(z)$, $f(z) = \ln(z)$ etc. we have to use the results from the beginning of this lecture.

- evaluating $f(A)$:

$$f(A) = u^{-1}f(\tilde{A})u.$$

Examples

Assume that the eigenvalues of A are parametrized by $\alpha_k = |\alpha_k|e^{i\theta_k}$. Below we also used the formulae from the first part of this lecture.

$$A^{-1} = u^{-1}\text{diag}\left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_n}\right) u$$

$$\begin{aligned}
\sqrt{A} &= u^{-1}\text{diag}\,(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots, \sqrt{\alpha_n})u \\
&= u^{-1}\text{diag}\,\left(\sqrt{|\alpha_1|}e^{i\theta_1/2}, \sqrt{|\alpha_2|}e^{i\theta_2/2}, \dots, \sqrt{|\alpha_n|}e^{i\theta_n/2}\right) u
\end{aligned}$$

$$\begin{aligned}
A^{-1/2} &= u^{-1}\text{diag}\,\left(\frac{1}{\sqrt{\alpha_1}}, \frac{1}{\sqrt{\alpha_2}}, \dots, \frac{1}{\sqrt{\alpha_n}}\right) u \\
&= u^{-1}\text{diag}\,\left(\frac{1}{\sqrt{|\alpha_1|}}e^{-i\theta_1/2}, \frac{1}{\sqrt{|\alpha_2|}}e^{-i\theta_2/2}, \dots, \frac{1}{\sqrt{|\alpha_n|}}e^{-i\theta_n/2}\right) u
\end{aligned}$$

$$\begin{aligned}
\sqrt[3]{A} &= u^{-1}\text{diag}\,(\sqrt[3]{\alpha_1}, \sqrt[3]{\alpha_2}, \dots, \sqrt[3]{\alpha_n})u \\
&= u^{-1}\text{diag}\,\left(\sqrt[3]{|\alpha_1|}e^{i\theta_1/3}, \sqrt[3]{|\alpha_2|}e^{i\theta_2/3}, \dots, \sqrt[3]{|\alpha_n|}e^{i\theta_n/3}\right) u
\end{aligned}$$

$$\exp A = u^{-1}\mathrm{diag}\,(\exp\alpha_1, \exp\alpha_2, \ldots, \exp\alpha_n)u$$

$$\begin{aligned}\ln A &= u^{-1}\mathrm{diag}\,(\ln(\alpha_1), \ln(\alpha_2), \ldots, \ln(\alpha_n))u \\ &= u^{-1}\mathrm{diag}\;(\ln|\alpha_1| + i\theta_1, \ln|\alpha_2| + i\theta_2, \ldots, \ln|\alpha_1| + i\theta_n)\,u\end{aligned}$$

Assume that A is 4×4 diagonal matrix with eigenvalues

$$\alpha_1 = 4, \quad \alpha_2 = 5(1+i), \quad \alpha_3 = 6(1+i\sqrt{3}), \quad \alpha_4 = -\frac{1}{4}.$$

Calculate $A^{2/3}$, $A^{-1/3}$, $\ln A$.

Next Lecture is on

Tensor products of Pauli matrices

Basis for parametrizing quantum bits:

Tensor product of Pauli matrices:

$$\sigma_k \otimes \sigma_m, \qquad k, m = 0, 1, 2, 3;$$

$$\sigma_0 \otimes \sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \qquad \sigma_1 \otimes \sigma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

$$\sigma_2 \otimes \sigma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}, \qquad \sigma_3 \otimes \sigma_k = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix},$$