Lecture 11

ELEMENTARY QUANTUM CIRCUITS AND PROGRAMS

of the course "Fundamentals of Quantum Computing"

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План

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$$U = U_L \dots U_1$$

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where each gate $\in^{\P} \mathrm{U}(H^{\otimes n})$.
If the initial (pure) state is Ψ then the final state will be $U^* \Psi U$.
If the initial mixed state is ρ then the final state will be $U^* \rho U$
This circuit is called *plain circuit*.

Definition 2 (ancilla)

Let the initial state is $\psi = \{|\psi_0\rangle, \ldots, |\psi_{n-1}\rangle\}$ and the final state is $\varphi = \{|\varphi_0\rangle, \ldots, |\varphi_{n-1}\rangle\}$. The plain circuit U transforming ψ to φ can be "enriched" by auxiliary qubits $\omega_0^{(i)}, \ldots, \omega_{\omega-1}^{(i)}$ which are specific for to this circuit.

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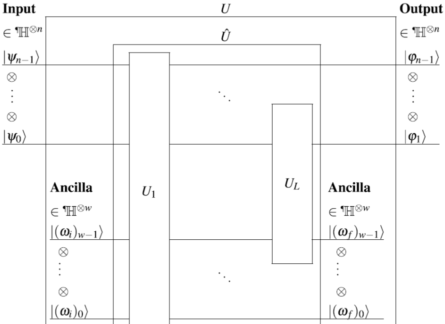
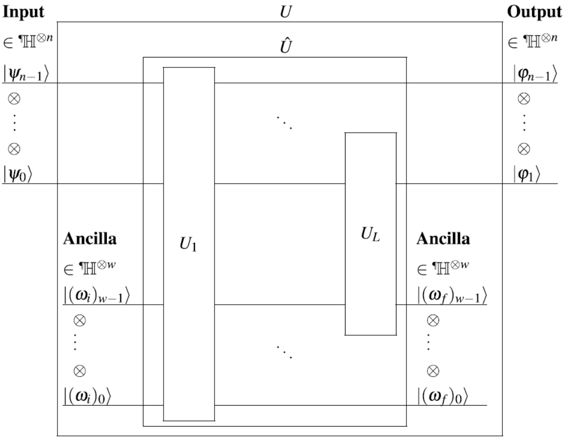
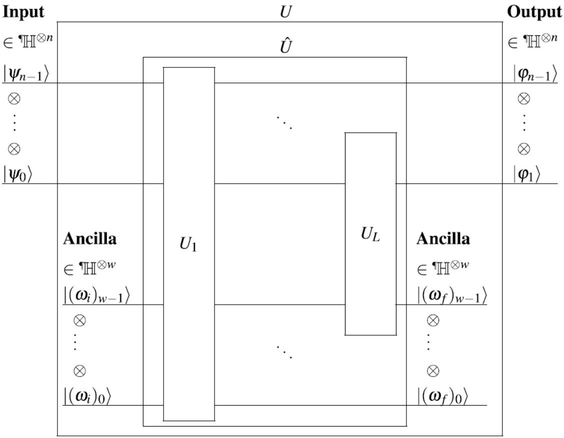




- the auxillary registers are not read at the end
- the auxillary registers should not be allowed to entangle with ψ
- if the input density is ρ then the output is again $U\rho U^*$



- the auxillary registers are not read at the end
- the auxillary registers should not be allowed to entangle with ψ
- if the input density is ρ then the output is again $U\rho U^*$ and can be expressed as $U\rho U^* = \mathrm{Tr}(\hat{U}(\rho \otimes \rho_{\omega^{(i)}})\hat{U}^*)$

Program 1. Quantum adder

Add numbers using digits


```
    10001111111101
+   00011100111110
   ---------------
=   ??????????????
```

Add numbers using digits

$$a = \sum_{j=0}^{n-1} a_j 2^j \,, \qquad b = \sum_{j=0}^{n-1} b_j 2^j$$

$$a + b = \sum_{j=0}^{n-1} s_j 2^j + c_n^+ 2^n \,,$$

$$\begin{aligned} c_j^+ &:= a_{j-1} b_{j-1} \overset{2}{\oplus} a_{j-1} c_{j-1}^+ \overset{2}{\oplus} b_{j-1} c_{j-1}^+ \qquad \text{for } j \in \{1, \ldots, n\} \\ s_j &:= a_j \overset{2}{\oplus} b_j \overset{2}{\oplus} c_j^+ \qquad \text{for } j \in \{0, \ldots, n-1\} \end{aligned}$$

$$U_3 U_2 U_1 |\Psi[b,a,0]\rangle = |c_n^+\rangle \otimes \bigotimes_{l=n-1}^{0} \left(|s_l\rangle \otimes |a_l\rangle \otimes |0\rangle\right)$$

$$\hat{U}_+ = U_0^* U_3 U_2 U_1 U_0$$

$$U_0\Big(|b\rangle \otimes |a\rangle \otimes |w\rangle\Big) := |b_n\rangle \otimes \bigotimes_{l=n-1}^{0} \left(|b_l\rangle \otimes |a_l\rangle \otimes |w_l\rangle\right)$$

$$=: |\Psi[b,a,w]\rangle$$

Theorem 5.38 *There exists a circuit U_+ on $\mathbb{H}^{I/O} = \mathbb{H}^B \otimes \mathbb{H}^A$, which can be implemented with the help of the auxiliary register $\mathbb{H}^W$ by $\hat{U}_+$, that is, there exists a $\hat{U}_+ \in \mathcal{U}(\mathbb{H}^{I/O} \otimes \mathbb{H}^W)$ such that for arbitrary $|\Phi\rangle \in \mathbb{H}^{I/O}$ one has*

$$\hat{U}_+ \left(|\Phi\rangle \otimes |0\rangle^n\right) = \left(U_+|\Phi\rangle\right) \otimes |0\rangle^n . \tag{5.101}$$

Furthermore, for $a, b \in \mathbb{N}_0$ with $a, b < 2^n$ we have that

$$U_3 U_2 U_1 |\Psi[b, a, 0]\rangle = |\Psi[b + a, a, 0]\rangle \tag{5.102}$$

and thus

$$U_+ \left(|b\rangle \otimes |a\rangle\right) = |b + a\rangle \otimes |a\rangle . \tag{5.103}$$

“Building” gates - U_s and U_c

$$
\begin{aligned}
A &:= \mathbf{1}^{\otimes 3} + (X - \mathbf{1}) \otimes |1\rangle\langle 1| \otimes \mathbf{1} \\
B &:= \mathbf{1}^{\otimes 3} + (X - \mathbf{1}) \otimes \mathbf{1} \otimes |1\rangle\langle 1| \\
U_s &:= BA
\end{aligned}
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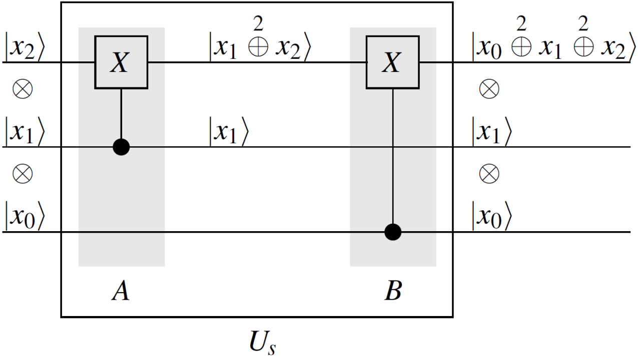

“Building” gates - U_s and U_c

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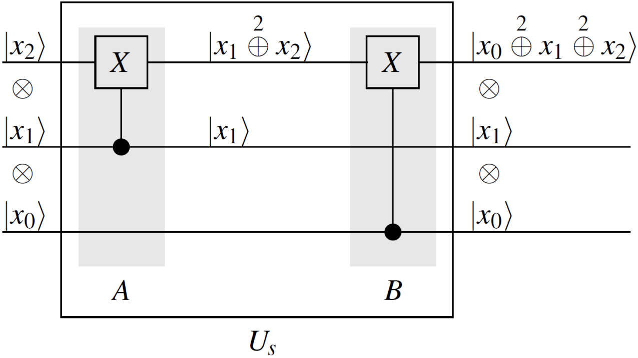

$$U_s\Big(|b_j\rangle \otimes |a_j\rangle \otimes |c_j^+\rangle\Big) = |s_j\rangle \otimes |a_j\rangle \otimes |c_j^+\rangle$$

“Building” gates - U_s and U_c

$$
\begin{aligned}
C &:= \mathbf{1}^{\otimes 4} + (X - \mathbf{1}) \otimes |1\rangle\langle 1| \otimes |1\rangle\langle 1| \otimes \mathbf{1} \\
D &:= \mathbf{1}^{\otimes 4} + \mathbf{1} \otimes (X - \mathbf{1}) \otimes |1\rangle\langle 1| \otimes \mathbf{1} \\
E &:= \mathbf{1}^{\otimes 4} + (X - \mathbf{1}) \otimes |1\rangle\langle 1| \otimes \mathbf{1} \otimes |1\rangle\langle 1| \\
U_c &:= EDC\,.
\end{aligned}
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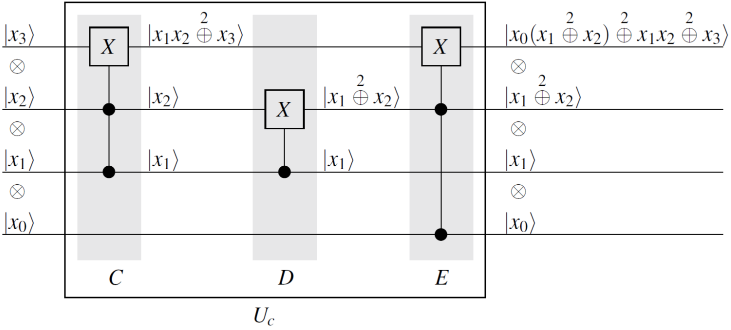

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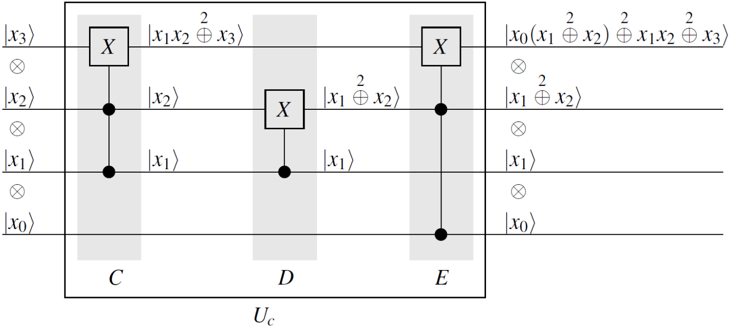

$$U_c\Big(|0\rangle \otimes |b_{j-1}\rangle \otimes |a_{j-1}\rangle \otimes |c^+_{j-1}\rangle\Big)$$

$$= |c^+_j\rangle \otimes |b_{j-1} \overset{2}{\oplus} a_{j-1}\rangle \otimes |a_{j-1}\rangle \otimes |c^+_{j-1}\rangle$$

U_1, U_2, U_3

$$U_1 := \prod_{l=1}^{n-1} \left(\mathbf{1}^{\otimes 3l} \otimes U_c \otimes \mathbf{1}^{\otimes 3(n-1-l)} \right)$$

$$U_2 := \left[\left(\mathbf{1} \otimes U_s \right) \left(\mathbf{1} \otimes \Lambda_{|1\rangle^1}(X) \otimes \mathbf{1} \right) U_c \right] \otimes \mathbf{1}^{\otimes 3(n-1)}$$

$$U_3 := \prod_{l=n-1}^{1} \left(\mathbf{1}^{\otimes 3l} \otimes \left(\mathbf{1} \otimes U_s \right) U_c^* \otimes \mathbf{1}^{\otimes 3(n-1-l)} \right)$$

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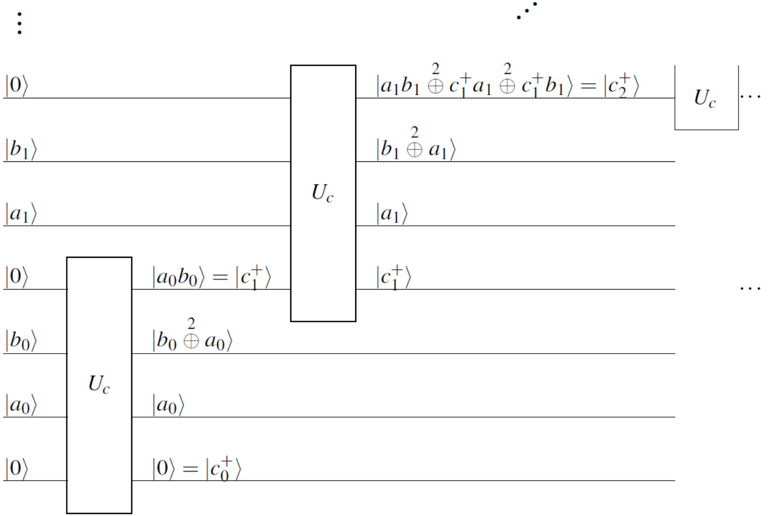
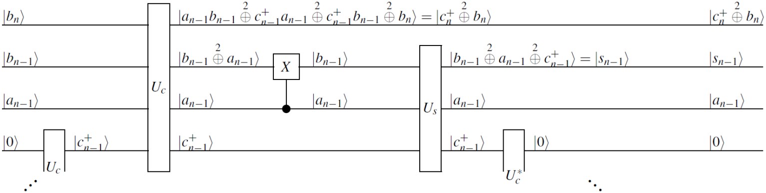


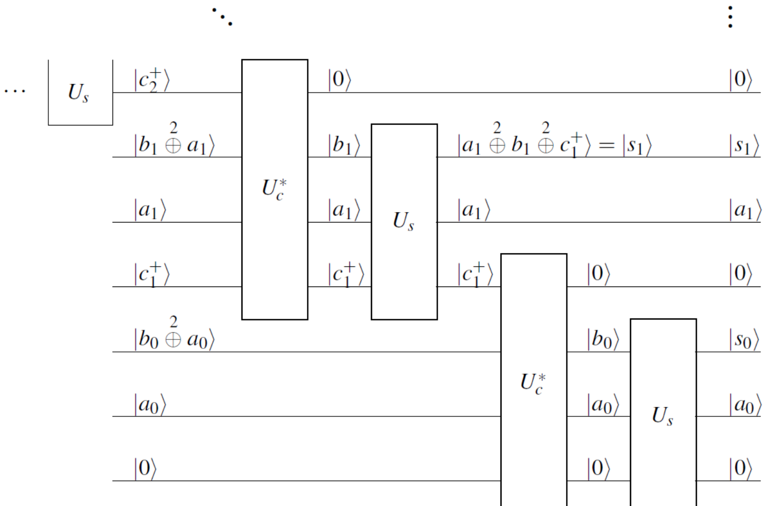
U_2

$$U_2 := \left[\left(\mathbf{1}\otimes U_s\right)\left(\mathbf{1}\otimes \Lambda_{|1\rangle^1}(X)\otimes \mathbf{1}\right)U_c\right]\otimes \mathbf{1}^{\otimes 3(n-1)}$$

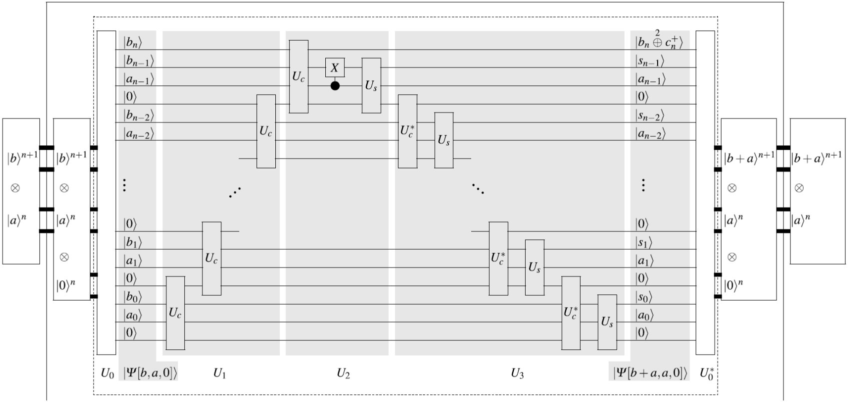


U_3

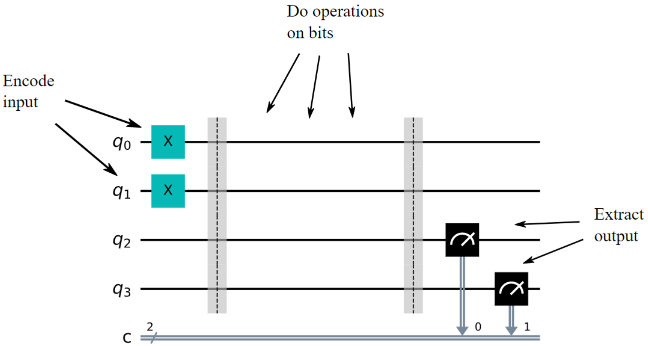
$$U_3 := \prod_{l=n-1}^{1} \left(\mathbf{1}^{\otimes 3l} \otimes (\mathbf{1} \otimes U_s) U_c^* \otimes \mathbf{1}^{\otimes 3(n-1-l)} \right)$$



$$
\begin{aligned}
\hat{U}_+\left(|b\rangle \otimes |a\rangle \otimes |0\rangle^n\right) &= U_0^* U_3 U_2 U_1 |\Psi[b,a,0]\rangle \\
&= U_0^* |\Psi[b+a-c_{n+1}^+ 2^{n+1}, a, 0]\rangle \\
&= |b+a-c_{n+1}^+ 2^{n+1}\rangle \otimes |a\rangle \otimes |0\rangle^n \\
&= U_+\left(|b\rangle \otimes |a\rangle\right) \otimes |0\rangle^n .
\end{aligned}
$$

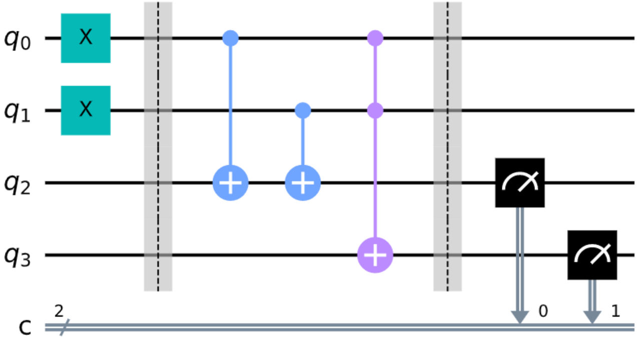


Half adder in Qiskit



https://qiskit.org/textbook/ch-states/atoms-computation.html#4.3-Adding-with-Qiskit-

Half adder in Qiskit




```
qc_ha.cx(0,2)
qc_ha.cx(1,2)
qc_ha.ccx(0,1,3)
qc_ha.measure(2,0)
qc_ha.measure(3,1)
```

Homework

Write qiskit code to calculate $4 + 3$ and submit the code to `quantumcomputing@quanterall.com`

THANK YOU FOR YOUR ATTENTION!

БЛАГОДАРЯ ЗА ВНИМАНИЕТО!