Lecture 13

ELEMENTARY QUANTUM CIRCUITS AND PROGRAMS. PART 3 - QUANTUM FOURIER TRANSFORM

of the course "Fundamentals of Quantum Computing"

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План

Fourier transform (refresher)

Quantum Fourier transform (QFT)

QFT in Qiskit

Fourier transform (refresher)

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The trigonometric Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

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The Fourier series in complex form is

$$\sum_{n=-\infty}^{\infty} c_n e^{-inx} \tag{1}$$

with

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \tag{2}$$

Fourier transform (refresher)

In QM, if we make a Fourier transform of the wave function

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we see that $\Psi(r)$ can be seen as a "sum" of plain waves (with good momentum) each with the amplitude $\Phi(p)/\left((2\pi\hbar)^{3/2}\right)$.

Fourier transform (refresher)

In signal processing Fourier transform relates the time and frequency picture

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Eases differenciation and intregration:

$$\partial_t f(t) \leftarrow \mathcal{F} \rightarrow i\omega F(\omega) \tag{3}$$

$$\int_{-\infty}^{t} f(\tau)d\tau \leftarrow \mathcal{F} \rightarrow \frac{1}{i\omega}F(\omega) + \pi F(0)\delta(\omega) \tag{4}$$

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The convolution of two function equals the product of their Fourier images

$$h(t) * x(t) \leftarrow \mathcal{F} \rightarrow H(\omega)X(\omega)$$

Discrete Fourier transform (DFT)

$$F(\omega) = \int f(t) e^{-i\omega t} dt$$

$$H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}, h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}$$

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Fast Fourier Transfor (Danielson и Lanczos (1942))

$$F_k = \sum_{j=0}^{N-1} W^{ij} f_j = ... = F_k^e + W^k F_k^o$$

$O(n \log n)$

Fourier transform (refresher)

$$\begin{pmatrix} F[0] \\ F[1] \\ F[2] \\ \vdots \\ F[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & W & W^2 & W^3 & \dots & W^{N-1} \\ 1 & W^2 & W^4 & W^6 & \dots & W^{N-2} \\ 1 & W^3 & W^6 & W^9 & \dots & W^{N-3} \\ \vdots & & & & & \\ 1 & W^{N-1} & W^{N-2} & W^{N-3} & \dots & W \end{pmatrix} \begin{pmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{pmatrix}$$

where $W = \exp(-j2\pi/N)$ and $W = W^{2N}$ etc. $= 1$.

Quantum Fourier transform (QFT)

$$F := \frac{1}{\sqrt{2^n}} \sum_{x,y=0}^{2^n-1} \exp\left(2\pi \mathrm{i} \frac{xy}{2^n}\right) |x\rangle \langle y|$$

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Our goal is to express it in a circuit using two-three gates.

Let us have a mixed state

$$|\boldsymbol{\Psi}\rangle = \sum_{x=0}^{2^n-1} \boldsymbol{\Psi}_x |x\rangle \quad \in \P\mathbb{H}^{\otimes n}$$

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$$|\Psi\rangle = \sum_{x=0}^{2^n-1} \Psi_x |x\rangle \quad \in \P\mathbb{H}^{\otimes n}$$

The components of its Fourier transform $F\Psi$ are expressed as the components of the discrete Fourier transform of $\{\Psi_x\}$

$$\langle k|F\Psi\rangle = F_{dis}(\mathbf{c})_k$$

Let us have a pure state x and expore the action of F on it

Lemma 5.52 *Let $n \in \mathbb{N}$ and*

$$x = \sum_{j=0}^{n-1} x_j 2^j , \tag{5.126}$$

where $x_j \in \{0, 1, \}$ for $j \in \{0, \ldots, n-1\}$.

Then the action of the quantum FOURIER *transform F on any vector $|x\rangle$ of the computational basis of $^{¶}\mathbb{H}^n$ can be written as*

$$F|x\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{j=0}^{n-1} \left[|0\rangle + \mathrm{e}^{2\pi \mathrm{i} 0.x_j \ldots x_0} |1\rangle \right] . \tag{5.127}$$

with

$$0.a_1 a_2 \ldots a_m := \frac{a_1}{2} + \frac{a_2}{4} + \cdots \frac{a_m}{2^m} = \sum_{l=1}^{m} a_l 2^{-l}$$

Lemma 5.53 *Let $n \in \mathbb{N}$ and $j \in \mathbb{N}_0$ with $j < n$ and let $|x\rangle$ be a vector in the computational basis in ¶$\mathbb{H}^{\otimes n}$. Then*

$$H|x_j\rangle = \frac{|0\rangle + e^{2\pi i 0.x_j}|1\rangle}{\sqrt{2}}, \tag{5.130}$$

holds, and with

$$H_j := \mathbf{1}^{\otimes(n-1-j)} \otimes H \otimes \mathbf{1}^{\otimes j} \tag{5.131}$$

for $j \in \{0, \ldots, n-1\}$ one has

$$H_j|x\rangle = |x_{n-1}\rangle \otimes \cdots |x_{j+1}\rangle \otimes \frac{|0\rangle + e^{2\pi i 0.x_j}|1\rangle}{\sqrt{2}} \otimes |x_{j-1}\rangle \otimes \cdots \otimes |x_0\rangle. \tag{5.132}$$

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H_j is the first gate we'll need to construct a QFT circuit

The second gate we'll need to construct a QFT circuit is called conditional phase shift

Definition 5.54 Let $j,k \in \{0,\ldots,n-1\}$ with $j > k$ and define $\theta_{jk} := \frac{\pi}{2^{j-k}}$. The **conditional phase shift** is a linear transformation on ¶$\mathbb{H}^{\otimes n}$ defined as

$$P_{jk} := \mathbf{1}^{\otimes(n-1-k)} \otimes |0\rangle\langle 0| \otimes \mathbf{1}^{\otimes k} + \mathbf{1}^{\otimes(n-1-j)} \otimes \Big[|0\rangle\langle 0| + \mathrm{e}^{\mathrm{i}\theta_{jk}} |1\rangle\langle 1| \Big] \otimes \mathbf{1}^{j-k-1} \otimes |1\rangle\langle 1| \otimes \mathbf{1}^{\otimes k}.$$

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$$\begin{aligned} P_{jk} &:= \mathbf{1}^{\otimes(n-1-k)} \otimes |0\rangle\langle 0| \otimes \mathbf{1}^{\otimes k} \\ &\quad + \mathbf{1}^{\otimes(n-1-j)} \otimes \Big[|0\rangle\langle 0| + \mathrm{e}^{\mathrm{i}\theta_{jk}}|1\rangle\langle 1|\Big] \otimes \mathbf{1}^{j-k-1} \otimes |1\rangle\langle 1| \otimes \mathbf{1}^{\otimes k}. \end{aligned}$$

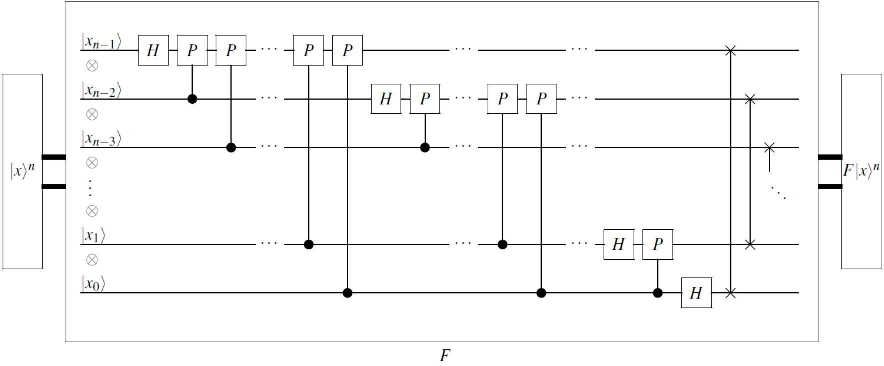
Lemma 5.55 *Let $j, k \in \{0, \ldots, n-1\}$ with $j > k$ and let $l \in \{j+1, \ldots, n-1\}$. Moreover, let $|\psi_l\rangle \in {}^{\P}\mathbb{H}$ and $\psi_{0j}, \psi_{1j} \in \mathbb{C}$ as well as $x_0, \ldots, x_{j-1} \in \{0, 1\}$. Then we have*

$$P_{jk}|\psi_{n-1}\rangle \otimes \cdots \otimes |\psi_{j+1}\rangle \otimes \big[\psi_{0j}|0\rangle + \psi_{1j}|1\rangle\big] \otimes |x_{j-1}\rangle \otimes \cdots \otimes |x_0\rangle$$

$$= |\psi_{n-1}\rangle \otimes \cdots \otimes |\psi_{j+1}\rangle \otimes \big[\psi_{0j}|0\rangle + \psi_{1j}\mathrm{e}^{\mathrm{i}\pi\frac{x_k}{2^{j-k}}}|1\rangle\big] \otimes |x_{j-1}\rangle \otimes \cdots \otimes |x_0\rangle$$

The QFT circuit is then:

$$F = S^{(n)} \prod_{j=0}^{n-1} \left(\left[\prod_{k=0}^{j-1} P_{jk} \right] H_j \right)$$
$$= S^{(n)} H_0 P_{1,0} H_1 P_{2,0} P_{2,1} H_2 \ldots P_{n-1,0} \ldots P_{n-1,n-2} H_{n-1}$$



QFT in Qiskit

https://qiskit.org/textbook/ch-algorithms/quantum-fourier-transform.html

THANK YOU FOR YOUR ATTENTION!

БЛАГОДАРЯ ЗА ВНИМАНИЕТО!