INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME

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1. **Integrable spinor models:**

   - Nambu–Jona-Lasinio–Vaks–Larkin models (1961); related to $SU(n)$
   - Gross–Neveu models (1974); related to $SP(2n)$
     proposed as models for describing the strong interactions.
   - Zakharov – Mikhailov spinor models related to $SO(n)$. In two-dimensions these models are integrable by (ISM) (Zakharov, Mikhailov, (1980).

2. **Construction of their solutions**

i) Nambu-Jona-Lasinio-Vaks-Larkin models. Choose $\mathfrak{g} \simeq su(N)$

$$\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{2a} \psi_\alpha \sum_{\beta=1}^{N} \psi_\beta^* \phi_\beta, \qquad \frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{2a} \phi_\alpha \sum_{\beta=1}^{N} \phi_\beta^* \psi_\beta.$$

ii) Gross-Nevew models. Choose $\mathfrak{g} \simeq sp(2N, \mathbb{R})$; with the standard definition of symplectic group elements:

$$\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{a} \psi_\alpha \sum_{\beta=1}^{N} (\psi_\beta \phi_\beta^* - \psi_\beta^* \phi_\beta), \qquad \frac{\partial \psi_\alpha}{\partial \xi} = -\frac{i}{a} \phi_\alpha \sum_{\beta=1}^{N} (\phi_\beta \psi_\beta^* - \phi_\beta^* \psi_\beta).$$

iii) Zakharov–Mikhailov models. Choose $\mathfrak{g} \simeq so(N, \mathbb{R})$; then $\psi(\xi, \eta)$ and $\phi(\xi, \eta)$ take values in the group $\mathfrak{G} \simeq SO(N, \mathbb{R})$.

$$i \frac{\partial \psi_\alpha}{\partial \xi} = \sum_{\beta=1}^{N} (\phi_\alpha \phi_\beta^* - \phi_\alpha^* \phi_\beta) \psi_\beta, \qquad i \frac{\partial \phi_\alpha}{\partial \eta} = \sum_{\beta=1}^{N} (\psi_\alpha \psi_\beta^* - \psi_\alpha^* \psi_\beta) \phi_\beta,$$

1 References

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2 The Lie algebras and Lie groups

The unitary $SU(N)$ and $su(N)$

$$\phi = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,N} \end{pmatrix} \in SU(N) \qquad \text{iff} \qquad \phi^\dagger = \phi^{-1}$$

$$U_1 \in su(N) \qquad \text{iff} \qquad U_1 = -U_1^\dagger; \qquad V_1 \in su(N) \qquad \text{iff} \qquad V_1 = -U_1^\dagger.$$

The symplectic $SP(2N)$ and $sp(2N)$

$$\phi = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \in SP(2N) \qquad \text{iff} \qquad S_0 \phi^T S_0^{-1} = \phi^{-1}, \quad S_0 = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}.$$

$$U_1 \in sp(2N) \qquad \text{iff} \qquad U_1^T + S_0 U_1 S_0^{-1} = 0.$$

$$\phi^{-1} = \begin{pmatrix} \boldsymbol{d}^T & -\boldsymbol{b}^T \\ -\boldsymbol{c}^T & \boldsymbol{a}^T \end{pmatrix}, \qquad U_1 = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & -\boldsymbol{A}^T \end{pmatrix}, \quad \boldsymbol{B} = \boldsymbol{B}^T, \quad \boldsymbol{C} = \boldsymbol{C}^T.$$

The orthogonal $SO(N)$ and $so(N)$

$$\phi = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,N} \\ \phi_{2,1} & \phi_{2,2} & \cdots & \phi_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N,1} & \phi_{N,2} & \cdots & \phi_{N,N} \end{pmatrix} \in SO(N) \qquad \text{iff} \qquad \phi^T = \phi^{-1}$$

$$U_1 \in so(N) \qquad \text{iff} \qquad U_1 = -U_1^T; \qquad V_1 \in so(N) \qquad \text{iff} \qquad V_1 = -U_1^T.$$

3 Lax representations:

$$\Psi_\xi = U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \qquad \Psi_\eta = V(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda),$$

$$U(\xi,\eta,\lambda) = \frac{U_1(\xi,\eta)}{\lambda - a}, \qquad V(\xi,\eta,\lambda) = \frac{V_1(\xi,\eta)}{\lambda + a},$$

where $\eta = t + x$, $\xi = t - x$ and a is a real number. Impose $\mathbb{Z}_2$-reduction:

$$U^\dagger(x,t,\lambda) = -U(x,t,\lambda^*), \qquad V^\dagger(x,t,\lambda) = -V(x,t,\lambda^*).$$

The compatibility condition of the linear problems reads:

$$\Psi_{\xi,\eta} \equiv \Psi_{\eta,\xi},$$

$$\Psi_{\xi,\eta} = U_\eta\Psi + U\Psi_\eta = (U_\eta + UV)\Psi, \qquad \Psi_{\eta,\xi} = V_\xi\Psi + V\Psi_\xi = (V_\xi + VU)\Psi,$$

$$U_\eta - V_\xi + [U,V] = 0,$$

i.e.

$$\frac{U_{1,\eta}}{\lambda - a} - \frac{V_{1,\xi}}{\lambda + a} + \frac{[U_1,V_1]}{(\lambda - a)(\lambda + a)} = 0,$$

$$\frac{U_{1,\eta}}{\lambda - a} - \frac{V_{1,\xi}}{\lambda + a} + \frac{[U_1, V_1]}{(\lambda - a)(\lambda + a)} = 0,$$

1) Multiply by $\lambda - a$ and take $\lim_{\lambda \to a}$: 2) Multiply by $\lambda + a$ and take $\lim_{\lambda \to -a}$:

$$1) \quad U_{1,\eta} + \frac{1}{2a}[U_1, V_1] = 0, \qquad 2) \quad V_{1,\xi} - \frac{1}{2a}[V_1, U_1] = 0,$$

Choose $U_1, V_1 \in su(N)$, i.e. $U_1 = -U_1^\dagger$, $V_1 = -V_1^\dagger$

$$\phi, \psi \in SU(N), \qquad \text{i.e.} \qquad \phi^\dagger = \phi^{-1}.$$

$$U_1(\xi, \eta) = -i\phi J_0 \phi^\dagger, \qquad V_1(\xi, \eta) = i\psi I_0 \psi^\dagger, \qquad J_0 = -I_0 = \mathrm{diag}\,(1, 0, 0, \cdots, 0).$$

$$U_{1,\eta} + \frac{1}{2a}[U_1, V_1(\xi, \eta)] = 0, \qquad V_{1,\xi} - \frac{1}{2a}[V_1, U_1(\xi, \eta)] = 0.$$

$$U_{1,\eta} + \frac{1}{2a}[U_1, V_1(\xi,\eta)] = 0, \qquad V_{1,\xi} - \frac{1}{2a}[V_1, U_1(\xi,\eta)] = 0,$$

$$U_1(\xi,\eta) = -i\phi J_0 \phi^\dagger, \qquad V_1(\xi,\eta) = i\psi I_0 \psi^\dagger,$$

$$\begin{aligned} & -i\phi_\eta J_0 \phi^\dagger - \phi J_0 \phi_\eta^\dagger + \frac{i}{2a} V_1 \phi J_0 \phi^\dagger - \frac{i}{2a}\phi J_0 \phi^\dagger V_1 \\ &= i\left(-\phi_\eta + \frac{1}{2a}V_1\phi\right) J_0\phi^\dagger - i\phi J_0 \left(\phi_\eta^\dagger + \frac{1}{2a}\phi^\dagger V_1\right) = 0, \end{aligned} \tag{1}$$

i.e. $-\phi_\eta + \frac{1}{2a}V_1\phi = 0 \qquad - \phi_\eta^\dagger + \frac{1}{2a}\phi^\dagger V_1^\dagger = 0.$

$$\begin{aligned} & i\psi_\xi J_0 \psi^\dagger + i\psi J_0 \psi_\xi^\dagger + \frac{i}{2a} U_1 \psi J_0 \psi^\dagger - \frac{i}{2a}\psi J_0 \psi^\dagger U_1 \\ &= i\left(\psi_\xi + \frac{1}{2a}U_1\psi\right) J_0\psi^\dagger + i\psi J_0 \left(\psi_\xi^\dagger - \frac{1}{2a}\psi^\dagger U_1\right) = 0, \end{aligned} \tag{2}$$

i.e. $\psi_\xi + \frac{1}{2a}U_1\psi = 0 \qquad \psi_\xi^\dagger + \frac{1}{2a}\psi^\dagger U_1^\dagger = 0.$

Remember, since $J_0 = \mathrm{diag}\,(1, 0, 0, \ldots, 0)$. U_1 and V_1 acquire the form of direct product:

$$U_1(\xi,\eta) = -i\vec{\phi}\vec{\phi}^{\,\dagger} = -i \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} (\phi_1^*, \phi_2^*, \ldots, \phi_N^*),$$

$$V_1(\xi,\eta) = i\vec{\psi}\vec{\psi}^{\,\dagger} = i \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} (\psi_1^*, \psi_2^*, \ldots, \psi_N^*),$$

$$-\vec{\phi}_\eta + \frac{1}{2a} V_1 \vec{\phi} = 0, \qquad \vec{\psi}_\xi + \frac{1}{2a} U_1 \vec{\psi} = 0,$$

i) Nambu-Jona-Lasinio-Vaks-Larkin models.

$$\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{2a} \psi_\alpha \sum_{\beta=1}^{N} \psi_\beta^* \phi_\beta, \qquad \frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{2a} \phi_\alpha \sum_{\beta=1}^{N} \phi_\beta^* \psi_\beta.$$

$$A_{\mathrm{NJLVL}}$$
$$= \int_{-\infty}^{\infty} dx\, dt\, \left(i \sum_{\alpha=1}^{N} \left(\phi_\alpha^* \frac{\partial \phi_\alpha}{\partial \eta} + \psi_\alpha^* \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left| \sum_{\alpha=1}^{N} (\psi_\alpha^* \phi_\alpha) \right|^2 \right). \quad (3)$$

ii) Gross-Neveu models. Choose $\mathfrak{g} \simeq sp(2N, \mathbb{R})$; with the standard definition of symplectic group elements:

$$\psi^{-1}(\xi,\eta) = S_0 \psi^T(\xi,\eta) S_0^{-1}, \quad \hat{\phi}(\xi,\eta) = S_0 \phi^T(\xi,\eta) S_0^{-1}, \quad S_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Then the corresponding group and Lie algebraic elements become:

$$\phi = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}, \qquad \psi = \begin{pmatrix} \boldsymbol{u} & \boldsymbol{s} \\ \boldsymbol{v} & \boldsymbol{t} \end{pmatrix},$$
$$\phi^{-1} = \begin{pmatrix} \boldsymbol{d}^T & -\boldsymbol{b}^T \\ -\boldsymbol{c}^T & \boldsymbol{a}^T \end{pmatrix}, \qquad \psi^{-1} = \begin{pmatrix} \boldsymbol{t}^T & -\boldsymbol{s}^T \\ -\boldsymbol{v}^T & \boldsymbol{u}^T \end{pmatrix},$$

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$ are arbitrary real $N \times N$ matrices. Next we choose

$$U_1(\xi,\eta) = \phi J \phi^{-1}, \qquad V_1(\xi,\eta) = \psi J \psi^{-1}, \qquad J = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix},$$

$$B_0 = \mathrm{diag}\,(1, 0, \ldots, 0, 0).$$

Next we introduce the vectors corresponding to the first columns of $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\vec{a} = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{N,1} \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} \phi_{N+1,1} \\ \vdots \\ \phi_{2N,1} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} \psi_{1,1} \\ \vdots \\ \psi_{N,1} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \psi_{N+1,1} \\ \vdots \\ \psi_{2N,1} \end{pmatrix},$$

Then U_1 and V_1 take the form:

$$U_1 = \begin{pmatrix} -\vec{a}\vec{c}^T & \vec{a}\vec{a}^T \\ \vec{c}\vec{c}^T & \vec{c}\vec{a}^T \end{pmatrix}, \qquad V_1 = \begin{pmatrix} -\vec{u}\vec{v}^T & \vec{u}\vec{u}^T \\ \vec{v}\vec{v}^T & \vec{v}\vec{u}^T \end{pmatrix},$$

The next step is to introduce the N-component complex vectors:

$$\vec{\phi}(\xi,\eta) = \frac{1}{2}(\vec{a} + i\vec{c}), \quad \vec{\psi}(\xi,\eta) = \frac{1}{2}(\vec{u} + i\vec{v})$$

Finally the Gross-Neveu model take the form:

$$\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{a}\psi_\alpha \sum_{\beta=1}^{N} (\psi_\beta \phi_\beta^* - \psi_\beta^* \phi_\beta), \qquad \frac{\partial \psi_\alpha}{\partial \xi} = -\frac{i}{a}\phi_\alpha \sum_{\beta=1}^{N} (\phi_\beta \psi_\beta^* - \phi_\beta^* \psi_\beta).$$

The functional of the action is:

$$A_{\mathrm{GN}} = \int_{-\infty}^{\infty} dx\, dt \, \left(i \sum_{\alpha=1}^{N} \left(\phi_\alpha^* \frac{\partial \phi_\alpha}{\partial \eta} + \psi_\alpha^* \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left(\sum_{\alpha=1}^{N} (\psi_\alpha^* \phi_\alpha - \phi_\alpha^* \psi_\alpha) \right)^2 \right).$$

iii) Zakharov–Mikhailov models. Choose $\mathfrak{g} \simeq so(N, \mathbb{R})$; then $\psi(\xi, \eta)$ and $\phi(\xi, \eta)$ take values in the group $\mathfrak{G} \simeq SO(N, \mathbb{R})$. Use the standard definition of orthogonal group elements:

$$\psi^{-1}(\xi, \eta) = \psi^T(\xi, \eta), \qquad \phi^{-1}(\xi, \eta) = \phi^T(\xi, \eta).$$

Next choose

$$J = E_{1,N} - E_{N,1} = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix},$$

where $(E_{kp})_{nm} = \delta_{kn}\delta_{pm}$. Then we introduce vectors corresponding to the first and the last columns $\phi_{(1)}, \phi_{(N)}, \psi_{(1)}, \psi_{(N)}$:

$$\vec{a} = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{N,1} \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} \phi_{1,N} \\ \vdots \\ \phi_{N,N} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} \psi_{1,1} \\ \vdots \\ \psi_{N,1} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \psi_{1,N} \\ \vdots \\ \psi_{N,N} \end{pmatrix},$$

Then we intorduce the complex vectors:

$$\vec{\phi} = \frac{1}{2}(\vec{a} + i\vec{c}), \qquad \vec{\psi} = \frac{1}{2}(\vec{u} + i\vec{v}).$$

i.e.

$$\phi_\alpha(\xi, \eta) = \frac{1}{2}(\phi^{(1)}_{\alpha,1} + i\phi^{(N)}_{\alpha,N}), \quad \psi_\alpha(\xi, \eta) = \frac{1}{2}(\psi^{(1)}_{\alpha,1} + i\psi^{(N)}_{\alpha,N})$$

Thus the ZM–system becomes:

$$i\frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{a}\sum_{\beta=1}^{N}(\phi_\alpha \phi^*_\beta - \phi^*_\alpha \phi_\beta)\psi_\beta), \qquad i\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{a}\sum_{\beta=1}^{N}(\psi_\alpha \psi^*_\beta - \psi^*_\alpha \psi_\beta)\phi_\beta,$$

This system has as action functional

$$A_{\mathrm{ZM}} = \int_{-\infty}^{\infty} dx\; dt\; \left(i\sum_{\alpha=1}^{N}\left(\phi^*_\alpha \frac{\partial \phi_\alpha}{\partial \eta} + \psi^*_\alpha \frac{\partial \psi_\alpha}{\partial \xi}\right) \right.$$
$$\left. - \frac{1}{2a}\left(\sum_{\alpha,\beta=1}^{N}(\phi^*_\alpha \phi_\beta - \phi^*_\beta \phi_\alpha)(\psi^*_\alpha \psi_\beta - \psi^*_\beta \psi_\alpha)\right)\right).$$