INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME. 2

Vladimir S. Gerdjikov

Institute for Nuclear Research and Nuclear Energy
and Institute of Mathematics and Informatics
Bulgarian Academy of Sciences, Sofia, Bulgaria
Institute for Advanced Physical Studies and Quanterall, Sofia, Bulgaria

- The inverse scattering method
- The scattering matrix and its η-dependence
- The fundamental analytic solutions and the Riemann-Hilbert problem

Lax representations:

$$\Psi_\xi = \frac{U_1(\xi,\eta)}{\lambda - a}\Psi(\xi,\eta,\lambda), \qquad \Psi_\eta = \frac{V_1(\xi,\eta)}{\lambda + a}\Psi(\xi,\eta,\lambda),$$

Impose $\mathbb{Z}_2$-reduction:

$$U^\dagger(x,t,\lambda) = -U(x,t,\lambda^*), \qquad V^\dagger(x,t,\lambda) = -V(x,t,\lambda^*).$$

$$U_1(\xi,\eta) = -i\vec{\phi}\vec{\phi}^{\,\dagger} = -i\phi J_0\phi^\dagger, \qquad V_1(\xi,\eta) = -i\vec{\psi}\vec{\psi}^{\,\dagger} = -i\psi J_0\psi^\dagger,$$

Change:

$$U_1 \to U_1 = -i\phi J\phi^\dagger \qquad V_1 \to V_1 = -i\psi J\psi^\dagger.$$

$$J_0 = \mathrm{diag}\,(1,0,0,\ldots,0) \to J = J_0 - \frac{1}{N}\mathbb{1}_N = \frac{1}{N}\mathrm{diag}\,(N-1,-1,-1,\ldots,-1).$$

Compatibility condition:

$$\frac{U_{1,\eta}}{\lambda - a} - \frac{V_{1,\xi}}{\lambda + a} + \frac{[U_1, V_1]}{(\lambda - a)(\lambda + a)} = 0.$$

Indeed:

$$\phi J_0 \phi^\dagger - \phi J \phi^\dagger = \phi(J_0 - J)\phi^\dagger = \frac{1}{N}\phi \mathbb{1}_N \phi^\dagger = \frac{1}{N}\mathbb{1}_N.$$

$$\psi J_0 \psi^\dagger - \psi J \psi^\dagger = \psi(J_0 - J)\psi^\dagger = \frac{1}{N}\psi \mathbb{1}_N \psi^\dagger = \frac{1}{N}\mathbb{1}_N.$$

Change also:

$$\xi \to x, \qquad \eta \to t.$$

The scattering problem for L

$$L: \quad i\Psi_x = \frac{U_0(x,t)}{\lambda - a}\Psi(x,t,\lambda), \qquad M: \quad i\Psi_t = \frac{V_0(x,t)}{\lambda + a}\Psi(x,t,\lambda),$$
$$U_0 = \phi J \phi^\dagger, \qquad V_0 = \psi J \psi^\dagger.$$

In fact there is indeterminacy in the second operator:

$$M: \quad i\Psi_t = \frac{V_0(x,t)}{\lambda + a}\Psi(x,t,\lambda) - \Psi(x,t,\lambda)C(\lambda),$$

where $C(\lambda)$ will be determined below.

Boundary conditions, i.e. the limits of U_0 and V_0 for $x \to \pm\infty$. For the spinor models the natural boundary conditions are

$$\lim_{x\to\pm\infty} \psi(x,t) = \mathbb{1}_N, \qquad \lim_{x\to\pm\infty} \phi(x,t) = \mathbb{1}_N, \qquad \lim_{x\to\pm\infty} U_0(x) = J, \qquad \lim_{x\to\pm\infty} V_0(x) = J,$$

Asymptotic solutions:

$$i\Psi_{0,x} = \frac{J}{\lambda - a}\Psi_0(x,\lambda), \qquad \Psi_0(x,\lambda) = \exp\left(\frac{-iJx}{\lambda - a}\right).$$

Jost solutions:

$$\lim_{x\to\infty} \Psi(x,t,\lambda)\Psi_0^{-1}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \Phi(x,t)\Psi_0^{-1}(x,\lambda) = \mathbb{1}.$$

Integral equations for $\Phi(x,t,\lambda)$ and $\Psi(x,t,\lambda)$:

Scattering matrix and its t-dependence

$$\Phi(x,t,\lambda) = \Psi(x,t,\lambda)T(t,\lambda), \qquad T(t,\lambda) = \begin{pmatrix} a^-(\lambda) & -\vec{b}^{-,T}(\lambda,t) \\ \vec{b}^+(\lambda,t) & \boldsymbol{a}^+(\lambda) \end{pmatrix}.$$

Consider the limit for $x \to -\infty$:

$$i\frac{\partial\Phi}{\partial t} = \frac{V_0}{\lambda+a}\Phi(x,t,\lambda) - \Phi(x,t,\lambda)C(\lambda) \qquad x \to -\infty,$$

$$\Rightarrow \qquad i\frac{\partial\Psi_0}{\partial t} = \frac{J}{\lambda+a}\Psi_0(x,\lambda) - \Psi_0(x,\lambda)C(\lambda), \qquad C(\lambda) = \frac{J}{\lambda+a}.$$

because $\frac{\partial\Psi_0}{\partial t} = 0$! Next consider the limit for $x \to \infty$ with $\Phi = \Psi T$:

$$\Rightarrow \qquad i\Psi_0(x,\lambda)\frac{\partial(T)}{\partial t} = \frac{J}{\lambda+a}\Psi_0(x,\lambda)T(t,\lambda) - \Psi_0(x,\lambda)T(t,\lambda)C(\lambda).$$

Finally:

$$i\frac{\partial T}{\partial t} = \left[\frac{J}{\lambda+a}, T(t,\lambda)\right].$$

In components we get:

$$\frac{\partial a^-(\lambda)}{\partial t} = 0, \qquad \frac{\partial \boldsymbol{a}^+(\lambda)}{\partial t} = 0, \quad i\frac{\partial \vec{b}^-}{\partial t} = \frac{2\vec{b}^-(\lambda,t)}{\lambda+a}, \quad i\frac{\partial \vec{b}^+}{\partial t} = -\frac{2\vec{b}^+(\lambda,t)}{\lambda+a},$$

Thus $a^-(\lambda)$ and $\boldsymbol{a}^+(\lambda)$: i) are analytic functions of λ for $\operatorname{Im}\lambda < 0$ and $\operatorname{Im}\lambda > 0$; ii) provide generating functionals of conservation laws for the spinor models. Usually for other models we use:

$$\ln a^-(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^{-k}; \qquad \frac{\partial I_k}{\partial t} = 0.$$

and I_k come out to have densities, which are local in the dynamical variables; besides I_k are in involution, i.e. the Poisson brackets $\{I_k, I_m\} = 0$ In this case we need to check if I_k will be local or nonlocal in $\vec{\phi}$ and $\vec{\psi}$.

Besides we have a whole $(N-1)\times(N-1)$ matrix $\boldsymbol{a}^+(\lambda)$ that also generates integrals of motion. Each matrix element of $\boldsymbol{a}^+(\lambda)$ generates conservation laws, but in general we can not expect neither local densities, nor vanishing Poisson brackets between these integrals.

Idea for solving the spinor models

$$
\begin{array}{ccccccc}
\vec{\phi}(x,t=0) & \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow & \vec{\phi}(x,t) \\
 & & \mathrm{I}\big\downarrow & & \big\uparrow\mathrm{III} & & \\
 & & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & &
\end{array}
$$

Jost solutions – integral equations

$$\Psi(x,\lambda) = \Psi_0(x,\lambda) - \frac{i}{\lambda-a}\int_{\infty}^{x} dy\ \Psi_0(x-y,\lambda)(U_0(y)-J)\Psi(y,\lambda),$$

$$\Phi(x,\lambda) = \Psi_0(x,\lambda) - \frac{i}{\lambda-a}\int_{-\infty}^{x} dy\ \Psi_0(x-y,\lambda)(U_0(y)-J)\Phi(y,\lambda),$$

$$\Psi_0(x-y,\lambda) = \exp\left(-\frac{i(x-y)}{\lambda-a}\left(\frac{N-1}{N},-\frac{1}{N},-\frac{1}{N},\ldots,-\frac{1}{N}\right)\right)$$

$$\Psi_0(x-y,\lambda) = \exp\left(\mathrm{Im}\left(\frac{(x-y)}{N(\lambda-a)}\right)(N-1,-1,-1,\ldots,-1) + \textit{oscillating}\right)$$

Assume, that $\lim_{x\to\pm\infty} U_0(x) = J$**. Besides:**
If $\mathrm{Im}\,(\lambda - a) > 0$**, then** $\mathrm{Im}\,\frac{1}{\lambda - a} < 0$**. Then:**

- If $\mathrm{Im}\,(\lambda - a) = 0$, then $\Psi_0(x-y,\lambda)$ oscillates and $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ are well defined!

- Consider $\Psi(x,\lambda)$: we have $y > x$ and therefore $\Psi_{0;11}(x-y,\lambda)$ decreases for $\mathrm{Im}\,\frac{1}{\lambda-a} > 0$; $\Psi_{0;22}(x-y,\lambda),\ldots,\Psi_{0;NN}(x-y,\lambda)$ decrease for $\mathrm{Im}\,\frac{1}{\lambda-a} < 0$. Therefore analytic extensions for the columns are possible:

$$\Psi(x,\lambda) = \left(\Psi^{+}_{(1)}(x,\lambda), \Psi^{-}_{(2)}(x,\lambda), \ldots, \Psi^{-}_{(N)}(x,\lambda)\right) = \left(\Psi^{+}_{(1)}(x,\lambda), \vec{\Psi}^{-}(x,\lambda)\right)$$

- Consider $\Phi(x,\lambda)$: we have $y < x$ and now the situation is opposite:

$$\Phi(x,\lambda) = \left(\Phi^{-}_{(1)}(x,\lambda), \Phi^{+}_{(2)}(x,\lambda), \ldots, \Phi^{+}_{(N)}(x,\lambda)\right) = \left(\Phi^{-}_{(1)}(x,\lambda), \vec{\Phi}^{+}(x,\lambda)\right)$$

We will need also the inverse of the Jost solutions:

$$\hat{\Psi} = \begin{pmatrix} \hat{\Psi}_1^- \\ \hat{\Psi}_1^+ \\ \vdots \\ \hat{\Psi}_N^+ \end{pmatrix}, \qquad \hat{\Phi} = \begin{pmatrix} \hat{\Phi}_1^+ \\ \hat{\Phi}_1^- \\ \vdots \\ \hat{\Phi}_N^- \end{pmatrix},$$

and the inverse of the scattering matrix:

$$\hat{\Phi} = \hat{T}\hat{\Psi}, \qquad \hat{T} = \begin{pmatrix} c^+ & \vec{d}^{+,T} \\ -\vec{d}^- & \boldsymbol{c}^- \end{pmatrix}$$

Here c^+ and $\boldsymbol{c}^-$, just like a^- and $\boldsymbol{a}^+$, are analytic functions of λ for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$. They also generate integrals of motion.

Fundamental analytic solutions

It is easy: if

$$\Psi(x,\lambda) = \left(\Psi^{+}_{(1)}(x,\lambda), \Psi^{-}_{(2)}(x,\lambda), \dots, \Psi^{-}_{(N)}(x,\lambda)\right)$$

$$\Phi(x,\lambda) = \left(\Phi^{-}_{(1)}(x,\lambda), \Phi^{+}_{(2)}(x,\lambda), \dots, \Phi^{+}_{(N)}(x,\lambda)\right)$$

then

$$\chi^{+}(x,\lambda) = \left(\Psi^{+}_{(1)}(x,\lambda), \Phi^{+}_{(2)}(x,\lambda), \dots, \Phi^{+}_{(N)}(x,\lambda)\right)$$

$$\chi^{-}(x,\lambda) = \left(\Phi^{-}_{(1)}(x,\lambda), \Psi^{-}_{(2)}(x,\lambda), \dots, \Psi^{-}_{(N)}(x,\lambda)\right)$$

Any two fundamental solutions are linearly related. Therefore:

$$\left(\Phi^{-}_{(1)}(x,\lambda), \vec{\Phi}^{+}(x,\lambda)\right) = \left(\Psi^{+}_{(1)}(x,\lambda), \vec{\Psi}^{-}(x,\lambda)\right) \begin{pmatrix} a^{-} & -\vec{b}^{-,T} \\ \vec{b}^{+} & \boldsymbol{a}^{+} \end{pmatrix}$$

$$\left(\Psi^+_{(1)}(x,\lambda), \vec{\Psi}^-(x,\lambda)\right) = \left(\Phi^-_{(1)}(x,\lambda), \vec{\Phi}^+(x,\lambda)\right) \begin{pmatrix} c^+ & \vec{d}^{+,T} \\ -\vec{d}^- & \boldsymbol{c}^- \end{pmatrix}$$

$$\Phi^-_{(1)} = \Psi^+_{(1)} a^- + \vec{\Psi}^- \vec{b}^+, \qquad \Psi^+_{(1)} = \Phi^-_{(1)} c^+ - \vec{\Phi}^+ \vec{d}^-,$$

$$\vec{\Phi}^+ = -\Psi^+_{(1)} \vec{b}^{-,T} + \vec{\Psi}^- \boldsymbol{a}^+, \qquad \vec{\Psi}^- = \Phi^-_{(1)} \vec{d}^{+,T} + \vec{\Phi}^+ \boldsymbol{c}^-,$$

$$\frac{\Phi^-_{(1)}}{a^-} = \Psi^+_{(1)} + \vec{\Psi}^- \frac{\vec{b}^+}{a^-}, \qquad \vec{\Phi}^+ \hat{\boldsymbol{a}}^+ = -\Psi^+_{(1)} \vec{b}^{-,T} \hat{\boldsymbol{a}^+} + \vec{\Psi}^-,$$

$$\chi^+(x,\lambda) = \left(\Psi^+_{(1)}, \vec{\Phi}^+ \hat{\boldsymbol{a}}^+\right) = \left(\Psi^+_{(1)}, \vec{\Psi}^-\right) \begin{pmatrix} 1 & -\vec{b}^{-,T} \hat{\boldsymbol{a}}^+ \\ 0 & \mathbb{1} \end{pmatrix} = \Psi(x,t,\lambda) S^+(t,\lambda),$$

$$\chi^-(x,\lambda) = \left(\frac{\Phi^-_{(1)}}{a^-}, \vec{\Psi}^-\right) = \left(\Psi^+_{(1)}, \vec{\Psi}^-\right) \begin{pmatrix} 1 & 0 \\ \frac{\vec{b}^+}{a^-} & \mathbb{1} \end{pmatrix} = \Psi(x,t,\lambda) S^-(t,\lambda),$$

$$\chi^+(x,\lambda) = \chi^-(x,\lambda) G_0(t,\lambda), \qquad G_0(t,\lambda) = \hat{S}^-(t,\lambda) S^+(\lambda), \qquad \lambda \in \mathbb{R}$$

$$G_0(t,\lambda) = \begin{pmatrix} 1 & 0 \\ -\frac{\vec{b}^+}{a^-} & \mathbb{1} \end{pmatrix} \begin{pmatrix} 1 & -\vec{b}^{-,T} \hat{\boldsymbol{a}}^+ \\ 0 & \mathbb{1} \end{pmatrix}.$$

Remember: $\chi^\pm(x,\lambda)$ **satisfy the equation:**

$$i\frac{\partial \chi^\pm}{\partial x} = \frac{U_0(x)}{\lambda - a} \chi^\pm(x,\lambda). \tag{1}$$

The direct scattering problem for L **(1):**
Let $U_0(x) - J$ be smooth function of x (Schwartz-type). Given $U_0(x)$ construct the scattering matrix $T(\lambda)$.

Minimal set of scattering data

$$\mathcal{T}_1 \equiv \{\vec{\rho}^{+,T}(t,\lambda) = \vec{b}^{-,T} \hat{\boldsymbol{a}}^+, \quad \vec{\rho}^-(t,\lambda) = \frac{\vec{b}^+}{a^-}, \quad \lambda \in \mathbb{R}\}$$

$\vec{\rho}^{\pm}(t,\lambda)$ – **reflection coefficients.**

Theorem $\mathcal{T}_1$ allows one to reconstruct both the full scattering matrix $T(t,\lambda)$ and the corresponding potential $U_0(x,t)$.

The inverse scattering problem for L **(1):**
Given the scattering matrix $T(\lambda)$ recover $U_0(x)$.

The inverse scattering problem for L is equivalent to the following

Riemann–Hilbert problem

Introduce new fundamental analytic solutions:

$$\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,\lambda)\hat{\Psi}_0(x,\lambda).$$

Advantage: canonical normalization. If the Lax operator depends polynomially on λ people choose:

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}.$$

In our case we choose different normalization, namely:

$$\lim_{\lambda \to a} \xi^{\pm}(x,t,\lambda) = \mathbb{1}, \qquad a \in \mathbb{R}.$$

On the real axis of the complex λ-plane we have:

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda) G(x,t,\lambda), \qquad G(x,t,\lambda) = \Psi_0(x,\lambda) G_0(t,\lambda) \Psi_0^{-1}(x,\lambda). \tag{2}$$

Riemann-Hilbert problem: Given the sewing function $\Psi_0(x,\lambda)$ for $\lambda \in \mathbb{R}$ construct the fundamental analytic solutions $\xi^{+}(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_{+}$ and $\xi^{-}(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_{-}$ such that they satisfy the canonical normalization and eq. (2).

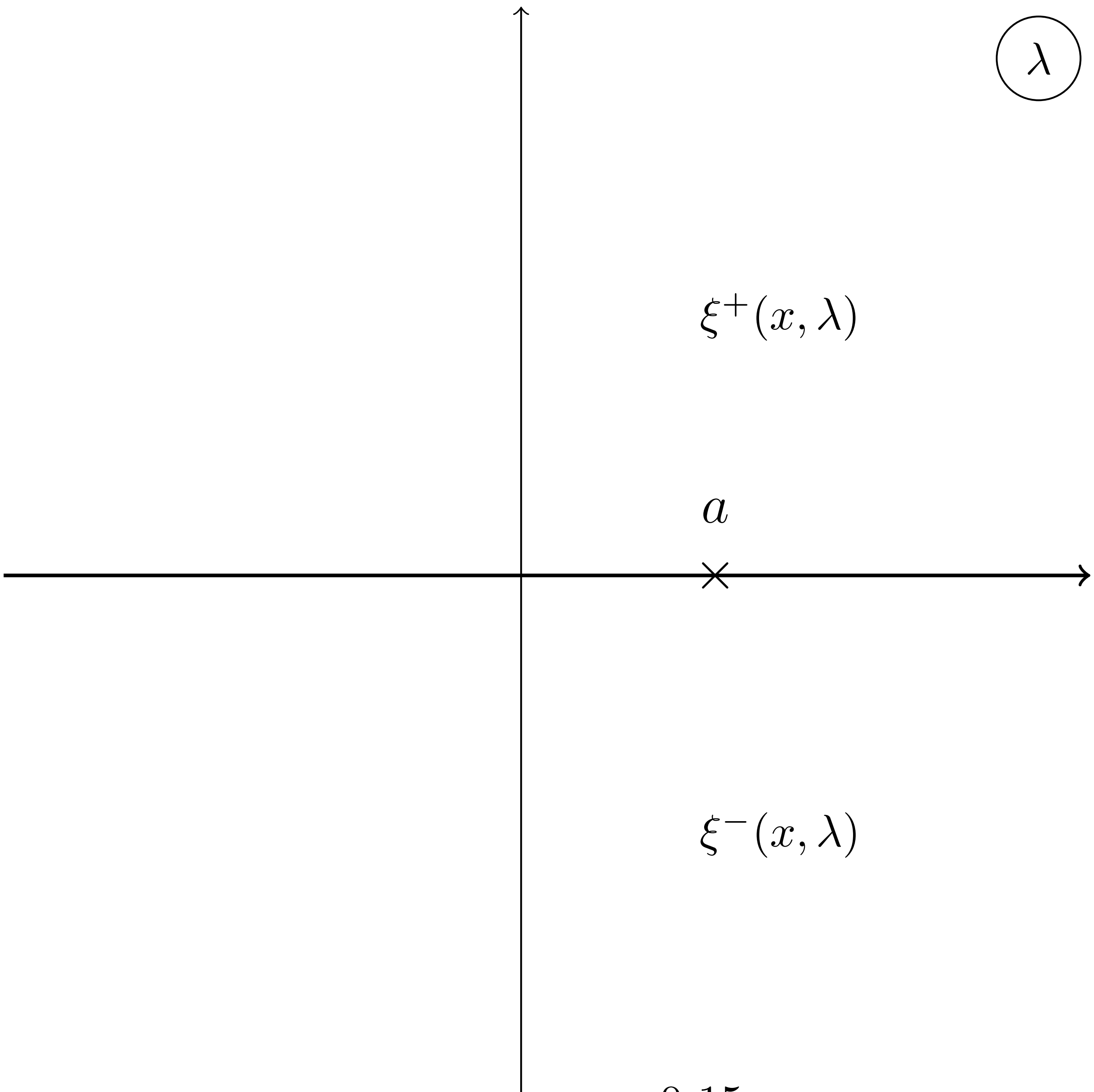
Remember: $\xi^{\pm}(x,\lambda)$ satisfy the equation:

$$i \frac{\partial \xi^{\pm}}{\partial x} = \frac{U_0(x)}{\lambda - a} \xi^{\pm}(x,\lambda) - \xi^{\pm}(x,\lambda) \frac{J}{\lambda - a}. \tag{3}$$

If we find the solution of the Riemann-Hilbert problem, the we can immediately find also $U_0(x)$. Multiply eq. (3) by $\lambda - a$ and by $\hat{\xi}^{\pm}(x,\lambda)$ on the right, then take the limit $\lambda \to a$:

$$\lim_{\lambda \to a} \quad : \quad i(\lambda - a)\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda) = U_0(x) - \xi^{\pm} J \hat{\xi}^{\pm}(x,\lambda),$$

$$\begin{aligned} U_0(x) &= \lim_{\lambda \to a}\left(\xi^{\pm} J \hat{\xi}^{\pm}(x,\lambda) - i(\lambda - a)\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right) \\ &= J - i\lim_{\lambda \to a}(\lambda - a)\left(\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right). \end{aligned} \tag{4}$$


λ
$\xi^+(x,\lambda)$
a
$\xi^-(x,\lambda)$