INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME. 3

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- Zakharov – Shabat dressing method and soliton solutions
- Soliton solutions of the Nambu–Jona-Lasinio–Vaks–Larkin model

The inverse scattering problem for L is equivalent to the following

Riemann–Hilbert problem

Introduce new fundamental analytic solutions:

$$\xi^{\pm}(x,t,\lambda) = \chi^{\pm}(x,\lambda)\hat{\Psi}_0(x,\lambda).$$

Advantage: canonical normalization. If the Lax operator depends polynomially on λ people choose:

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,t,\lambda) = \mathbb{1}.$$

In our case we choose different normalization, namely:

$$\lim_{\lambda\to a} \xi^{\pm}(x,t,\lambda) = \mathbb{1}, \qquad a \in \mathbb{R}.$$

On the real axis of the complex λ-plane we have:

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda)G(x,t,\lambda), \qquad G(x,t,\lambda) = \Psi_0(x,\lambda)G_0(t,\lambda)\Psi_0^{-1}(x,\lambda). \tag{1}$$

Riemann-Hilbert problem: Given the sewing function $\Psi_0(x,\lambda)$ for $\lambda \in \mathbb{R}$ construct the fundamental analytic solutions $\xi^+(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_+$ and $\xi^-(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_-$ such that they satisfy the canonical normalization and eq. (1).

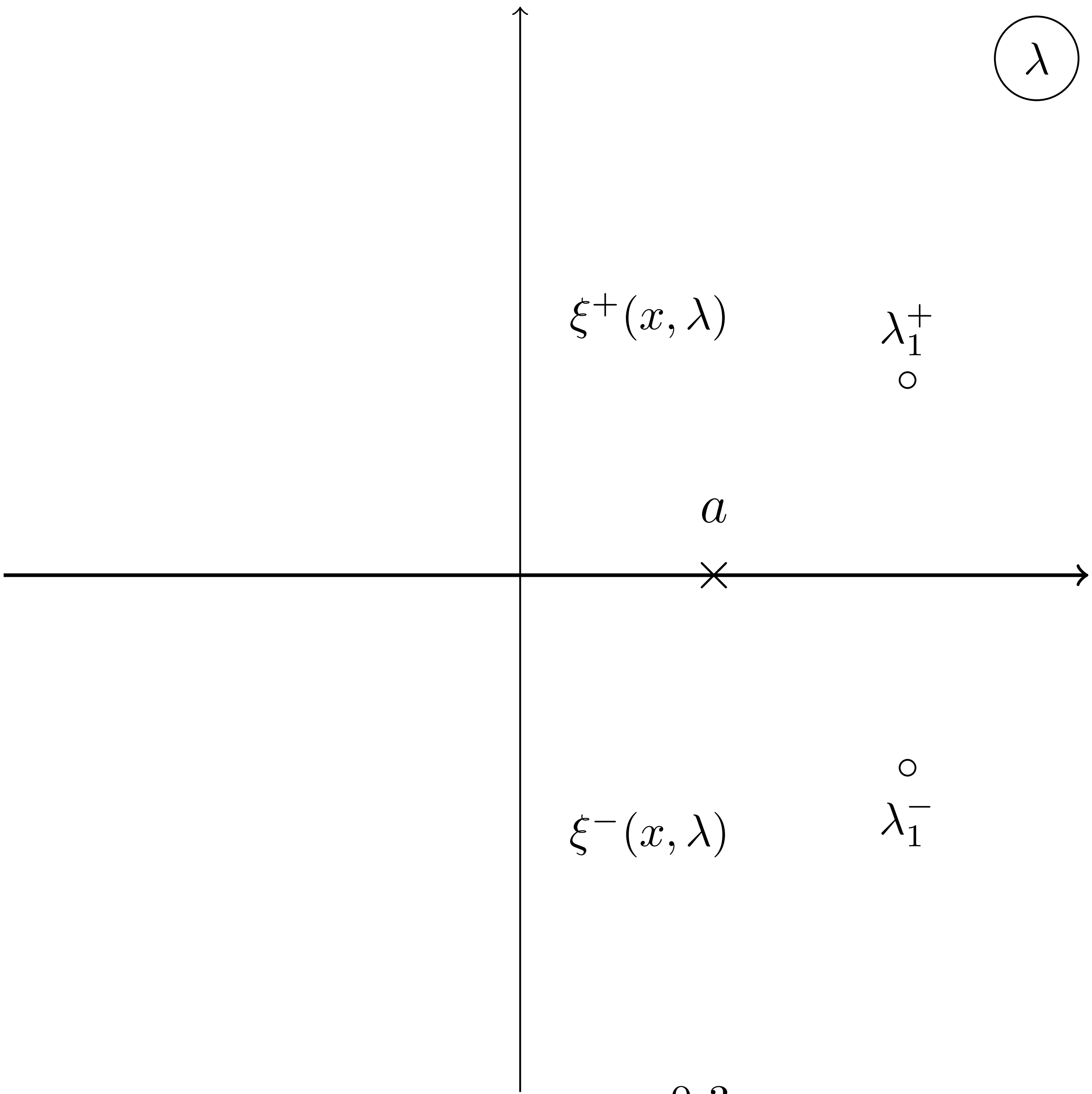
Remember: $\xi^\pm(x,\lambda)$ **satisfy the equation:**

$$i\frac{\partial \xi^\pm}{\partial x} = \frac{U_0(x)}{\lambda - a}\xi^\pm(x,\lambda) - \xi^\pm(x,\lambda)\frac{J}{\lambda - a}. \tag{2}$$

If we find the solution of the Riemann-Hilbert problem, the we can immediately find also $U_0(x)$. Multiply eq. (2) by $\lambda - a$ and by $\hat{\xi}^\pm(x,\lambda)$ on the right, then take the limit $\lambda \to a$:

$$\lim_{\lambda\to a} \quad : \quad i(\lambda - a)\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,\lambda) = U_0(x) - \xi^\pm J\hat{\xi}^\pm(x,\lambda),$$

$$\begin{aligned} U_0(x) &= \lim_{\lambda\to a}\left(\xi^\pm J\hat{\xi}^\pm(x,\lambda) - i(\lambda - a)\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,\lambda)\right) \\ &= J - i\lim_{\lambda\to a}(\lambda - a)\left(\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,\lambda)\right). \end{aligned} \tag{3}$$


λ
$\xi^+(x,\lambda)$
λ_1^+
a
$\xi^-(x,\lambda)$
λ_1^-

Riemann–Hilbert problem – canonical case

$$
\begin{aligned}
L\psi &\equiv \left(i\frac{\partial}{\partial x} + Q(x,t) - \lambda J \right) \psi(x,\lambda) = 0, \\
M\psi &\equiv \left(i\frac{\partial}{\partial t} + V_0(x,t) + \lambda V_1(x,t) - 2\lambda^2 J \right) \psi(x,\lambda) = 0, \\
Q(x,t) &= \left(\begin{array}{cc} 0 & \boldsymbol{q}(x) \\ \boldsymbol{r}(x) & 0 \end{array} \right), \qquad J = \left(\begin{array}{cc} 1 & 0 \\ 0 & -\mathbb{1} \end{array} \right),
\end{aligned}
\tag{4}
$$

where $Q(x,t)$ and J are $(n+1)\times(n+1)$ matrices with compatible block structure and $V_0(x,t)$, $V_1(x,t)$ are expressed in terms of Q and its x-derivative:

$$
V_1(x,t) = 2Q(x,t), \qquad V_0(x,t) = -[Q, \mathrm{ad}_J^{-1} Q] + 2i\,\mathrm{ad}_J^{-1} Q_x. \tag{5}
$$

$$
\mathrm{ad}_J Q = [J,Q] = \left(\begin{array}{cc} 0 & 2\boldsymbol{q} \\ -2\boldsymbol{r} & 0 \end{array} \right), \quad (\mathrm{ad}_J)^2 Q = [J,[J,Q]] = 4Q,
$$

$$
\Rightarrow \qquad (\mathrm{ad}_J)^{-1} = \frac{1}{4}\mathrm{ad}_J.
$$

ad_J has a kernel B: these are all block-diagonal matrices;

$$\mathrm{ad}_J B = [J, B] = 0, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix};$$

ad_J has an image R: these are all block-off-diagonal matrices R. ad_J can be inverted on its image:

$$\mathrm{ad}_J R = [J, R] \neq 0, \qquad R = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}, \qquad \mathrm{ad}_J^{-1} R = \begin{pmatrix} 0 & \frac{1}{2} R_1 \\ -\frac{1}{2} R_2 & 0 \end{pmatrix}.$$

Zakharov and Shabat (1971): scalar NLS $n = 1$ and $\mathfrak{g} \simeq su(2)$; Manakov (1974): vector NLS $n \geq 2$ and $\mathfrak{g} \simeq su(n)$;

Numerous physical applications in nonlinear optics, plasma physics, hydrodynamics etc. Review paper: VSG **ArXive: nlin.SI/0604004**.

The compatibility condition is the multicomponent NLS equation

$$\begin{aligned} i\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2(\boldsymbol{q}, \boldsymbol{r})\boldsymbol{q}(x,t) = 0, \\ -i\boldsymbol{r}_t + \boldsymbol{r}_{xx} + 2(\boldsymbol{r}, \boldsymbol{q})\boldsymbol{r}(x,t) = 0. \end{aligned} \tag{6}$$

Put $\boldsymbol{r} = \boldsymbol{q}^\dagger$ to get vector NLS:

$$i\boldsymbol{q}_t + \boldsymbol{q}_{xx} + 2(\boldsymbol{q}, \boldsymbol{q}^\dagger)\boldsymbol{q}(x,t) = 0, \tag{7}$$

The Riemann-Hilbert problem

Asymptotic Lax operator: $Q(x,t) \to 0$ for $x \to \pm\infty$;

$$L_0 \equiv i\frac{\partial \psi_0}{\partial x} - \lambda J \psi_0(x,\lambda) = 0, \qquad \psi_0(x,\lambda) = e^{-i\lambda J x}.$$

Jost solutions: $\lim_{x\to\infty} \Psi(x,\lambda) e^{i\lambda J x} = \mathbb{1}, \qquad \lim_{x\to-\infty} \Phi(x,\lambda) e^{i\lambda J x} = \mathbb{1},$

Fundamental analytic solutions They are constructed from the Jost solutions:

$$\Psi(x,\lambda) = \left(\Psi^-_{(1)}(x,\lambda), \Psi^+_{(2)}(x,\lambda), \dots, \Psi^+_{(N)}(x,\lambda)\right)$$

$$\Phi(x,\lambda) = \left(\Phi^+_{(1)}(x,\lambda), \Phi^-_{(2)}(x,\lambda), \dots, \Phi^-_{(N)}(x,\lambda)\right)$$

$$\chi^+(x,\lambda) = \left(\Phi^+_{(1)}(x,\lambda), \vec{\Psi}^+(x,\lambda)\hat{\boldsymbol{a}}^+\right)$$

$$\chi^-(x,\lambda) = \left(\Psi^-_{(1)}(x,\lambda)/a^-(\lambda), \vec{\Phi}^-(x,\lambda)\right)$$

The Jost solutions are linearly related by the scattering matrix $T(t,\lambda)$:

$$\left(\Phi^-_{(1)}(x,\lambda), \vec{\Phi}^+(x,\lambda)\right) = \left(\Psi^+_{(1)}(x,\lambda), \vec{\Psi}^-(x,\lambda)\right) T(t,\lambda)$$

$$\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda), \qquad \lambda \in \mathbb{R}.$$

Riemann-Hilbert problem:

$$\xi^\pm(x,\lambda) = \chi^\pm(x,\lambda)e^{i\lambda Jx}.$$

$$\xi^+(x,\lambda) = \xi^-(x,\lambda)G(x,\lambda), \qquad \lambda \in \mathbb{R}; \quad G(x,\lambda) \Rightarrow \text{sewing matrix.}$$

$$\lim_{\lambda\to\infty} \xi^\pm(x,\lambda) = \mathbb{1}, \qquad \text{canonical normalization.}$$

$$G(x,\lambda) = e^{-i\lambda Jx}G_0(\lambda)e^{i\lambda Jx}.$$

Definition: $\xi^{\pm}(x,\lambda)$ is a regular solution of RHP if $\det \xi^{\pm}(x,\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_{\pm}$.

Theorem: RHP with canonical normalization has unique regular solution.

Proof Let $\xi^{\pm}_{1,2}(x,\lambda)$ be two regular solutions of the same RHP. Then:

$$\xi_1^+(x,\lambda)\hat{\xi}_2^+(x,\lambda) = \xi_1^-(x,\lambda)G(x,\lambda)\hat{G}(x,\lambda)\hat{\xi}_2^-(x,\lambda) = \xi_1^-(x,\lambda)\hat{\xi}_2^-(x,\lambda),$$

$$\lim_{\lambda\to\infty} \xi_1^+(x,\lambda)\hat{\xi}_2^+(x,\lambda) = \mathbb{1}.$$

Liouville theorem: if the function $F(\lambda)$ is analytic for all $\lambda \in \mathbb{C}$ and has no singularities in λ then $F(\lambda) = \text{const}$.

$$\xi_1^+(x,\lambda)\hat{\xi}_2^+(x,\lambda) = \mathbb{1},$$

i.e.

$$\xi_1^+(x,\lambda) = \xi_2^+(x,\lambda).$$

Equivalence of RHP and Lax representations

Remember:

$$L\chi^{\pm}(x,\lambda) \equiv i\frac{\partial \chi^{\pm}}{\partial x} + (Q(x) - \lambda J)\chi^{\pm}(x,\lambda) = 0,$$

$$\mathcal{L}\xi^{\pm}(x,\lambda) \equiv i\frac{\partial \xi^{\pm}}{\partial x} + Q(x)\xi^{\pm}(x,\lambda) - \lambda[J, \xi^{\pm}(x,\lambda)] = 0,$$

Theorem [Zakharov, Shabat]. Let $\xi^{\pm}(x,\lambda)$ be solution to a RHP with canonical normalization and $G(x,t,\lambda)$ such that:

$$i\frac{dG}{dx} - \lambda[J, G(x,\lambda)] = 0,$$

Then

$$i\frac{d\xi^{\pm}}{dx} + Q(x)\xi^{\pm}(x,\lambda) - \lambda[J, \xi^{\pm}(x,\lambda)] = 0,$$

Proof:

$$g^{\pm}(x,\lambda) = i\frac{d\xi^{\pm}}{dx}\hat{\xi}^{\pm}(x,\lambda) + \lambda\xi^{\pm}(x,\lambda)J\hat{\xi}^{\pm}(x,\lambda).$$

$$
\begin{aligned}
g^{+}(x,\lambda) &= i\frac{d(\xi^{-}G)}{dx}\hat{G}\hat{\xi}^{-}(x,\lambda) + \lambda\xi^{-}GJ\hat{G}\hat{\xi}^{-}(x,\lambda) \\
&= i\frac{d\xi^{-}}{dx}\hat{\xi}^{-}(x,\lambda) + \xi^{-}\left(i\frac{dG}{dx}\hat{G} + \lambda GJ\hat{G}(x,\lambda)\right)\hat{\xi}^{-}(x,\lambda) \\
&= i\frac{d\xi^{-}}{dx}\hat{\xi}^{-}(x,\lambda) + \xi^{-}\left(\lambda[J,G]\hat{G} + \lambda GJ\hat{G}(x,\lambda)\right)\hat{\xi}^{-}(x,\lambda) \\
&= i\frac{d\xi^{-}}{dx}\hat{\xi}^{-}(x,\lambda) + \lambda\xi^{-}J\hat{\xi}^{-}(x,\lambda) \\
&\equiv g^{-}(x,\lambda), \qquad \lambda \in \mathbb{R}.
\end{aligned}
$$

Thus $g^{+}(x,\lambda) = g^{-}(x,\lambda)$ is analytic in the whole complex λ-plane except in the vicinity of $\lambda \to \infty$ where $g^{+}(x,\lambda)$ tends to λJ. Liouville theorem:

$$
g^{+}(x,\lambda) - \lambda J = \mathrm{const}
$$

with respect to λ; denote it $-q(x)$ and get:

$$
g^{+}(x,\lambda) - \lambda J = -Q(x).
$$

$$Q(x) = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm}(x,\lambda) J \hat{\xi}^{\pm}(x,\lambda) \right).$$

Similarly one treats also the time dependence.

Singular solutions of RHP and the soliton solutions of NLEE. Dressing method

The one-soliton solutions. Dressing factor $u_1(x,\lambda)$:

$$\xi_1^{\pm}(x,\lambda) = u_1(x,\lambda)\xi_0^{\pm}(x,\lambda)u_{1-}^{-1}(\lambda), \quad u_{1-} = \lim_{x\to-\infty} u_1(x,\lambda).$$

$$\xi_0^{\pm}(x,\lambda) = \chi_0^{\pm}(x,\lambda)e^{i\lambda J x}, \quad \xi_1^{\pm}(x,\lambda) = \chi_1^{\pm}(x,\lambda)e^{i\lambda J x}.$$

$$i\frac{d\chi_0^{\pm}}{dx} + Q_0(x)\chi_0^{\pm}(x,\lambda) - \lambda J \chi_0^{\pm}(x,\lambda) = 0,$$

$$i\frac{d\chi_1^{\pm}}{dx} + Q_1(x)\chi_1^{\pm}(x,\lambda) - \lambda J \chi_1^{\pm}(x,\lambda) = 0,$$

Thus the dressing factor satisfies:

$$i\frac{du_1}{dx} + Q_1(x)u_1(x,\lambda) - u_1(x,\lambda)Q_0(x) - \lambda[J, u_1(x,\lambda)] = 0,$$

Need an anzatz for u. Must be singular in λ.

$$u_1(x,\lambda) \quad = \quad \mathbb{1} + (c(\lambda) - 1)P_1(x), \qquad c(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-},$$

$P_1^2 = P_1$. If $\operatorname{rank} P_1 = 1$, then:

$$P_1(x) = \frac{|n_1\rangle\langle m_1|}{\langle m_1|n_1\rangle},$$

Insert into the equation for $u_1(x,\lambda)$ and request that it holds for $\lambda = \lambda_1^+$, $\lambda = \lambda_1^-$ and $\lambda \to \infty$. This allows us to express $|n_1\rangle$ and $\langle m_1|$ in terms of the **regular solution** χ_0 **only**.

$$|n_1\rangle = \chi_{01}^-(x,t)|n_{01}\rangle, \quad \langle m_1| = \langle m_{01}|\hat{\chi}_{01}^+(x,t), \quad \chi_{01}^\pm(x,t) = \chi_0^\pm(x,t,\lambda_1^\pm)$$

$$Q_1(x,t) \quad = \quad Q_0(x,t) - (\lambda_1^+ - \lambda_1^-)[J, P_1(x,t)].$$

One-soliton solutions of MNLS eqs. (6):

$$Q_0(x) = 0, \quad Q_1(x,t) = -(\lambda_1^+ - \lambda_1^-)[J, P_1(x,t)],$$

$$P_1(x) = \frac{|n_1\rangle\langle m_1|}{\langle m_1|n_1\rangle}, \quad |n_1\rangle = e^{-i(\lambda_1^- x + \lambda_1^{-2} t)J}|n_{10}\rangle, \quad \langle m_1| = \langle m_{10}|e^{i(\lambda_1^+ x + \lambda_1^{+2} t)J},$$

The two-soliton solutions of MNLS (6). Dressing factor $u_2(x,\lambda)$: Repeat the dressing starting from $\chi_1^\pm(x,t,\lambda)$:

$$\chi_2^\pm(x,t,\lambda) = u_2(x,t,\lambda)\chi_1^\pm(x,t,\lambda) = u_2(x,t,\lambda)\chi_1^\pm(x,t,\lambda),$$

Thus second dressing factor satisfies:

$$i\frac{du_2}{dx} + Q_2(x)u_2(x,\lambda) - u_2(x,\lambda)Q_1(x) - \lambda[J, u_2(x,\lambda)] = 0,$$

$$u_2(x,\lambda) \quad = \quad \mathbb{1} + (c_2(\lambda) - 1)P_2(x), \quad c_2(\lambda) = \frac{\lambda - \lambda_2^+}{\lambda - \lambda_2^-}, \quad P_2(x) = \frac{|n_2\rangle\langle m_2|}{\langle m_2|n_2\rangle},$$

Now we express $|n_2\rangle$ and $\langle m_2|$ in terms of the **regular solution** $\chi_1^{\pm}$ **only**.

$$|n_2\rangle = \chi_1^-(x,t,\lambda_2^-)|n_{02}\rangle, \quad \langle m_2| = \langle m_{02}|\hat{\chi}_1^+(x,t,\lambda_2^+), \quad \chi_1^{\pm}(x,t) = u_1(x,t,\lambda)\chi_0^{\pm}(x,t,\lambda$$

$$\begin{aligned} Q_2(x,t) &= Q_1(x,t) - (\lambda_2^+ - \lambda_2^-)[J, P_2(x,t)] \\ &= -(\lambda_1^+ - \lambda_1^-)[J, P_1(x,t)] - (\lambda_2^+ - \lambda_2^-)[J, P_2(x,t)]. \end{aligned}$$

In order to obtain the solution of the vector NLS (7) we need to impose the reduction:

$$\lambda_j^- = (\lambda_j^+)^*, \qquad \langle m_{0j}| = (|n_{0j}\rangle)^\dagger.$$

Spectral meaning of the dressing

- Each dressing procedure adds a pair of discrete eigenvalues λ_j^+ and λ_j^- to the spectrum of L.

- The projectors P_j project onto the discrete eigen-subspaces of L corresponding to the discrete eigenvalues $\lambda_j^{\pm}$. It may be of rank ≥ 1.

- Obviously the soliton solutions are rational functions of exponentials.

There are alternative methods for N-soliton solutions