INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME. 4

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- Lax representations and simple Lie algebras

1 Lax representations:

Example 1

$$\Psi_\xi = U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \qquad \Psi_\eta = V(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda),$$
$$U(\xi,\eta,\lambda) = \frac{U_1(\xi,\eta)}{\lambda - a}, \qquad V(\xi,\eta,\lambda) = \frac{V_1(\xi,\eta)}{\lambda + a},$$

where $\eta = t + x$, $\xi = t - x$ and a is a real number.

Example 2

$$L\psi \equiv \left(i\frac{\partial}{\partial x} + Q(x,t) - \lambda J \right) \psi(x,\lambda) = 0,$$
$$M\psi \equiv \left(i\frac{\partial}{\partial t} + V_0(x,t) + \lambda V_1(x,t) - 2\lambda^2 J \right) \psi(x,\lambda) = 0, \qquad (1)$$
$$Q(x,t) = \begin{pmatrix} 0 & \boldsymbol{q}(x) \\ \boldsymbol{r}(x) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

where $Q(x,t)$ and J are $(n+1)\times(n+1)$ matrices with compatible block structure and $V_0(x,t)$, $V_1(x,t)$ are expressed in terms of Q and its x-derivative:

$$V_1(x,t) = 2Q(x,t), \qquad V_0(x,t) = -[Q, \mathrm{ad}_J^{-1} Q] + 2i\mathrm{ad}_J^{-1} Q_x. \tag{2}$$

Example 3 Why not generalize, e.g.:

$$\begin{aligned} L\psi &\equiv \left(i\frac{\partial}{\partial x} + U_0(x,t) + \lambda U_1(x,t) \right) \psi(x,\lambda) = 0, \\ M\psi &\equiv \left(i\frac{\partial}{\partial t} + V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t) \right) \psi(x,\lambda) = 0, \end{aligned} \tag{3}$$

where $U_k(x,t)$ and $V_k(x,t)$ are $N \times N$ matrices taking values in a certain Lie algebra. The compatibility condition is rather complicated:

$$i\sum_{k=0}^{2} \lambda^k \frac{\partial V_k}{\partial x} - i\sum_{s=0}^{1} \lambda^s \frac{\partial U_s}{\partial t} + [U_0(x,t) + \lambda U_1(x,t), V_0(x,t) + \lambda V_1(x,t) + \lambda^2 V_2(x,t)] = 0.$$

How to simplify?

- Diagonalize U_1 and V_2

- Important constraint: the eigenvalues of U_1 and V_2 should be constants.
- All U_s and V_k must be elements of a simple Lie algebra.

Then we have chance to apply the inverse scattering method!

Idea for solving nonlinear evolution equations

$$
\begin{array}{ccccccc}
Q(x,t=0) & \longrightarrow & L_0 & & L|_{t>0} & \longrightarrow & Q(x,t) \\
 & & \mathrm{I}\big\downarrow & & \big\uparrow\mathrm{III} & & \\
 & & T(0,\lambda) & \xrightarrow{\mathrm{II}} & T(t,\lambda) & &
\end{array}
$$

Classes of equivalence of Lax operators

Fixing the gauge and gauge equivalence: Choose

$$\begin{aligned} L\psi &\equiv \left(i\frac{\partial}{\partial x} + Q(x,t) - \lambda J \right) \psi(x,\lambda) = 0, \\ M\psi &\equiv \left(i\frac{\partial}{\partial t} + V_0(x,t) + \lambda V_1(x,t) - 2\lambda^2 J \right) \psi(x,\lambda) = 0, \end{aligned} \tag{4}$$

where J is constant diagonal, and $Q(x,t)$ is generic element of $\mathfrak{g}$, e.g.:

$$Q(x,t) = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \qquad J = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Apply the gauge transformation:

$$\tilde{\psi}(x,t,\lambda) = g_0(x,t)\psi(x,t,\lambda), \qquad L \to \tilde{L} = g_0^{-1} L g_0(x,t), \qquad g_0 = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix},$$

$$\tilde{L}\tilde{\psi} = \frac{\partial \tilde{\psi}}{\partial x} + (-i g_{0,x} g_0^{-1} + g_0^{-1} Q(x,t) g_0(x,t) - \lambda J)\tilde{\psi} = 0.$$

Fix up g_0 by:

$$
\begin{aligned}
& - ig_{0,x}g_0^{-1} + g_0^{-1}Q^{\mathrm{d}}(x,t)g_0(x,t) \\
& = \left(\begin{array}{cc} -ig_{11,x}g_{11}^{-1} + g_{11}^{-1}q_{11}g_{11} & 0 \\ 0 & -ig_{22,x}g_{22}^{-1} + g_{22}^{-1}q_{22}g_{22} \end{array}\right) = 0.
\end{aligned} \tag{5}
$$

Then

$$
\tilde{L}\tilde{\psi} = \frac{\partial \tilde{\psi}}{\partial x} + (\tilde{Q} - \lambda J)\tilde{\psi} = 0, \qquad \tilde{Q} = g_0^{-1}Q^{\mathrm{off}}(x,t)g_0(x,t)
$$

i.e.

$$
\tilde{Q}(x,t) = \left(\begin{array}{cc} 0 & \tilde{q}_{12} \\ \tilde{q}_{21} & 0 \end{array}\right) = [J, Q_1(x,t)].
$$

Lax pair and simple Lie algebras

- Examples of simple Lie algebras
- Cartan–Weyl basis of $sp(4)$, $so(5)$, $sp(6)$, $so(7)$, $so(8)$;
- Every simple Lie algebra allows matrix representation!

Examples of simple Lie algebras

Four classes of simple Lie algebras by Cartan classification:
$sl(r+1)$, $so(2r+1)$, $sp(2r)$, $so(2r)$.

Special linear algebras $sl(r+1)$

$$sl(r+1), \qquad (r+1)\times(r+1) \text{ matrix} X \in sl(r+1). \quad \text{iff} \quad \operatorname{tr} X = 0.$$

Special orthogonal algebras $so(2r+1)$**,** $\operatorname{tr} X = 0$**.**

$$so(2r{+}1), \quad (2r{+}1)\times(2r{+}1) \text{ matrix} X \in so(2r{+}1). \quad \text{iff} \quad X + S_0 X^T S_0^{-1} = 0.$$

$$S_0 = \left(\begin{array}{c|c|c} 0_r & 0 & s_0 \\ \hline 0 & (-1)^r & 0 \\ \hline \hat{s}_0 & 0 & 0_r \end{array}\right) \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_0^2 = \mathbb{1}.$$

Special symplectic algebras $sp(2r)$

$$sp(2r), \qquad (2r)\times(2r) \text{ matrix} \quad X \in sp(2r). \quad \text{iff} \quad X + S_1 X^T S_1^{-1} = 0.$$

$$S_1 = \begin{pmatrix} 0 & s_1 \\ -\hat{s}_1 & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_1^2 = -\mathbb{1}$$

Special orthogonal algebras $so(2r)$

$$so(2r), \quad (2r)\times(2r) \text{ matrix} X \in so(2r). \quad \text{iff} \quad X + S_0 X^T S_0^{-1} = 0.$$

$$S_0 = \begin{pmatrix} 0 & s_0 \\ \hat{s}_0 & 0 \end{pmatrix}, \quad s_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0^2 = \mathbb{1}$$

Cartan subalgebra and Cartan-Weyl basis

Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$ is maximal commutative subalgebra of $\mathfrak{g}$
All elements of the Cartan subalgebra can be **simultaneously** diagonalized!
From now on all Cartan elements will be realized by diagonal matrices.

Cartan-Weyl basis of $A_2 \simeq sl(r+1)$

Let $r = 2$. Then X is 3×3 matrix. Generic element of $\mathfrak{h}$ is given by:

$$H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, \qquad \mathrm{tr}\, H = h_1 + h_2 + h_3 = 0.$$

Important: to every element $H \in \mathfrak{h} \Leftrightarrow \vec{h} \in \mathbb{E}_3$.

$$\vec{h} = (h_1, h_2, h_3)^T = \sum_{j=1}^{3} h_j \vec{e}_j, \qquad (\vec{h}, \vec{\epsilon}) = 0, \quad \vec{\epsilon} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3.$$

The **rank** of $sl(3)$ is $r = \dim(\mathfrak{h}) = 2$.

Weyl generators E_α: eigenvalues of ad_H:

$$H = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_H(E_\alpha) \equiv [H, E_\alpha] = (h_2 - h_3)E_\alpha = (\vec{h}, \vec{e}_2 - \vec{e}_3)E_\alpha,$$

$$\mathrm{ad}_H(E_\beta) \equiv [H, E_\beta] = (h_1 - h_2)E_\beta = (\vec{h}, \vec{e}_1 - \vec{e}_2)E_\beta,$$

Weyl generators are labeled by vectors α called roots. All possible roots provides the root system of $sl(3)$.

$$\alpha = \vec{e}_2 - \vec{e}_3, \qquad \beta = \vec{e}_1 - \vec{e}_2, \qquad \Delta_{A_2} \equiv \{\vec{e}_j - \vec{e}_k, \quad 1 \leq j \neq k \leq 3\}$$

Positive and negative roots:

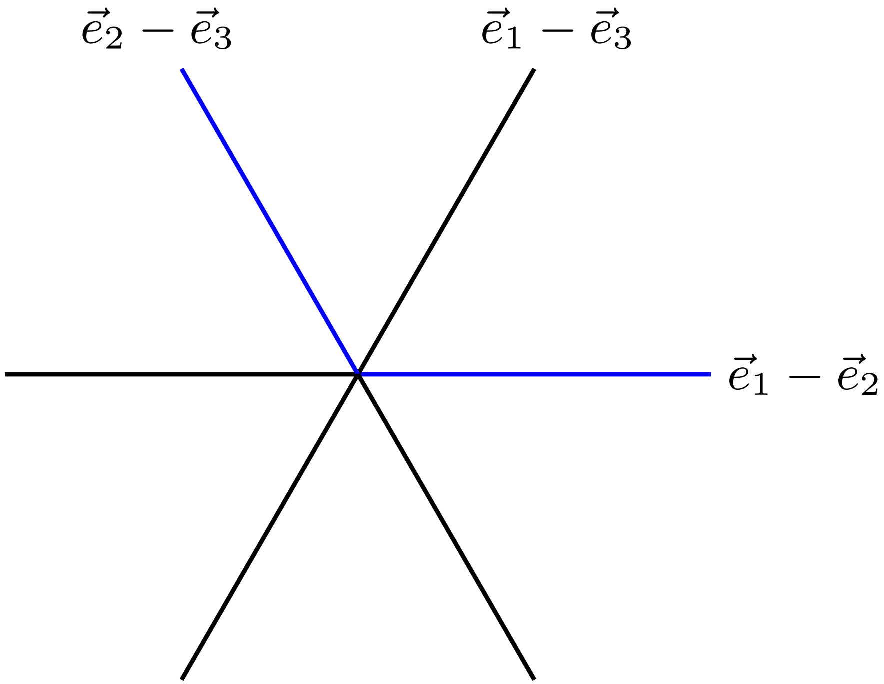
$$\Delta^+_{A_2} \equiv \{\vec{e}_1 - \vec{e}_2, \vec{e}_1 - \vec{e}_3, \vec{e}_2 - \vec{e}_3\}, \qquad \Delta^-_{A_2} \equiv \{-\vec{e}_1 + \vec{e}_2, -\vec{e}_1 + \vec{e}_3, -\vec{e}_2 + \vec{e}_3\}.$$

E_α with $\alpha > 0$ is upper-triangular; E_β with $\beta < 0$ is lower-triangular;

$$E_{-\alpha} = E_\alpha^T.$$

Note: all roots $\alpha \in \Delta_{A_2}$ satisfy $(\alpha, \vec{\epsilon}) = 0$

$$\dim(\Delta_{A_2}) = 2 = \text{rank}\, sl(3).$$



Cartan-Weyl basis for $sl(3)$ in the 3-dimensional representation:

$$H_k = E_{k,k} - E_{k+1,k+1}, \quad k = 1, 2; \qquad E_{e_j - e_k} = E_{j,k},$$

where

$$(E_{jk})_{mn} = \delta_{jm}\delta_{kn}.$$

Cartan-Weyl basis of $A_3 \simeq sl(r+1)$

Let $r = 3$. Then X is 4×4 matrix. Generic element of $\mathfrak{h}$ is given by:

$$H = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix}, \qquad \mathrm{tr}\, H = h_1 + h_2 + h_3 + h_4 = 0.$$

Important: to every element $H \in \mathfrak{h} \Leftrightarrow \vec{h} \in \mathbb{E}_4$.

$$\vec{h} = (h_1, h_2, h_3, h_4)^T = \sum_{j=1}^{4} h_j \vec{e}_j, \qquad (\vec{h}, \vec{\epsilon}) = 0, \quad \vec{\epsilon} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3 + \vec{e}_4.$$

The **rank** of $sl(4)$ is $r = \dim(\mathfrak{h}) = 3$.

Weyl generators E_α: eigenvalues of ad_H:

$$H = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_H(E_\alpha) \equiv [H, E_\alpha] = (h_2 - h_3)E_\alpha = (\vec{h}, \vec{e}_2 - \vec{e}_3)E_\alpha,$$

$$\mathrm{ad}_H(E_\beta) \equiv [H, E_\beta] = (h_1 - h_3)E_\beta = (\vec{h}, \vec{e}_1 - \vec{e}_3)E_\beta,$$

Weyl generators are labeled by vectors α called roots. All possible roots provides the root system of $sl(4)$.

$$\alpha = \vec{e}_2 - \vec{e}_3, \qquad \beta = \vec{e}_1 - \vec{e}_3, \qquad \Delta_{A_3} \equiv \{\vec{e}_j - \vec{e}_k, \quad 1 \le j \ne k \le 4\}$$

Positive and negative roots:

$$\Delta^+_{A_3} \equiv \{\vec{e}_j - \vec{e}_k, \quad 1 \le j < k \le 4\}, \ldots, \qquad \Delta^-_{A_3} \equiv \{\vec{e}_j - \vec{e}_k, \quad 4 \ge j > k \ge 1\}.$$

E_α with $\alpha > 0$ is upper-triangular; E_β with $\beta < 0$ is lower-triangular;

$$E_{-\alpha} = E_\alpha^T.$$

Note: all roots $\alpha \in \Delta_{A_3}$ satisfy $(\alpha, \vec{\epsilon}) = 0$

$$\dim(\Delta_{A_3}) = 3 = \mathrm{rank}\, sl(3).$$

Cartan-Weyl basis for $sl(4)$ in the 4-dimensional representation:

$$H_k = E_{k,k} - E_{k+1,k+1}, \quad k = 1, 2, 3; \qquad E_{e_j - e_k} = E_{j,k},$$

where

$$(E_{jk})_{mn} = \delta_{jm}\delta_{kn}.$$

Cartan-Weyl basis of $B_2 \simeq so(5)$

Let $r = 2$, so $2r + 1 = 5$. Then X is 5×5 matrix, such that

$$X + S_0 X^T S_0^{-1} = 0, \qquad S_0 = \left(\begin{array}{c|c|c} 0_2 & 0 & s_0 \\ \hline 0 & 1 & 0 \\ \hline \hat{s}_0 & 0 & 0_2 \end{array}\right), \qquad s_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Symmetry with respect to the second diagonal. Thus $\mathfrak{h}$ consists of diagonal matrices.

Generic element of $\mathfrak{h}$ is given by:

$$H = \left(\begin{array}{cc|c|cc} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & 0 & -h_1 \end{array}\right), \qquad \operatorname{tr} H = 0, \qquad \dim(\mathfrak{h}) = 2$$

Important: $H \in \mathfrak{h} \Leftrightarrow \vec{h} = (h_1, h_2)^T \in \mathbb{E}_2$. Then $\operatorname{rank} so(5) = 2$.

Weyl generators E_α: eigenvalues of ad_H:

$$E_\alpha = \left(\begin{array}{cc|c|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \ E_\beta = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad E_\gamma = \left(\begin{array}{cc|c|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$[H, E_\alpha] = (h_1 - h_2)E_\alpha = (\vec{h}, \vec{e}_1 - \vec{e}_2)E_\alpha, \qquad [H, E_\beta] = (h_1 + h_2)E_\beta = (\vec{h}, \vec{e}_1 + \vec{e}_2)E_\beta,$$

$$[H, E_\gamma] = h_1 E_\gamma = (\vec{h}, \vec{e}_1)E_\gamma,$$

Weyl generators are labeled by vectors α called roots. All possible roots provides the root system of $so(5)$. Positive roots

$$\Delta^+_{B_2} \equiv \{\vec{e}_1 - \vec{e}_2, \quad \vec{e}_1 + \vec{e}_2, \quad \vec{e}_1, \quad \vec{e}_2, \}$$

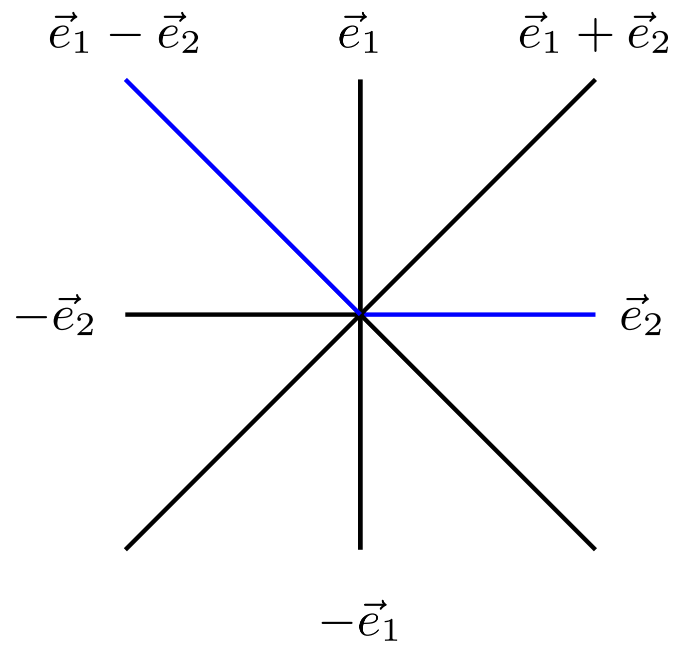
and negative roots:

$$\Delta^-_{B_2} \equiv \{-\vec{e}_1 + \vec{e}_2, \quad -\vec{e}_1 - \vec{e}_2, \quad -\vec{e}_1, \quad -\vec{e}_2\}.$$

E_α with $\alpha > 0$ is upper-triangular; E_β with $\beta < 0$ is lower-triangular;

$$E_{-\alpha} = E_\alpha^T.$$

Long roots $\pm\vec{e}_1 \pm \vec{e}_2$ with length 2; short roots $\pm\vec{e}_1$ and $\pm\vec{e}_2$ with length 1.



$B_3 \simeq so(7)$, rank $so(7) = 3$

$$\Delta^+_{B_3} \equiv \{\vec{e}_1 \pm \vec{e}_2, \quad \vec{e}_1 \pm \vec{e}_3, \quad \vec{e}_2 \pm \vec{e}_3, \quad \vec{e}_1, \quad \vec{e}_2, \quad \vec{e}_3, \}$$

Cartan-Weyl basis for $so(7)$ in the 7-dimensional representation:

$$E_{e_j-e_k} = E_{jk} - (-1)^{j+k} E_{8-k,8-j}, \qquad E_{e_j+e_k} = E_{j,8-k} - (-1)^{j+k} E_{k,8-j},$$

$$E_{e_j} = E_{j,4} - (-1)^j E_{4,8-j}.$$

Cartan-Weyl basis of $C_2 \simeq sp(4)$

Let $r = 2$. Then X is 4×4 matrix, such that

$$X + S_1 X^T S_1^{-1} = 0, \qquad S_1 = \left(\begin{array}{c|c} 0_2 & s_1 \\ \hline -\hat{s}_1 & 0_2 \end{array} \right), \qquad s_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_1^2 = -\mathbb{1}$$

Thus $\mathfrak{h}$ consists of diagonal matrices.

$$H = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & -h_1 \end{pmatrix}, \qquad \mathrm{tr}\, H = 0, \qquad \dim(\mathfrak{h}) = 2$$

Important: $H \in \mathfrak{h} \Leftrightarrow \vec{h} = (h_1, h_2)^T \in \mathbb{E}_2$. Then rank $sp(4) = 2$.

Weyl generators E_α: eigenvalues of ad_H:

$$E_\alpha = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad E_\beta = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad E_\gamma = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$[H, E_\alpha] = (h_1 - h_2) E_\alpha = (\vec{h}, \vec{e}_1 - \vec{e}_2) E_\alpha, \qquad [H, E_\beta] = (h_1 + h_2) E_\beta = (\vec{h}, \vec{e}_1 + \vec{e}_2) E_\beta,$$

$$[H, E_\gamma] = 2h_1 E_\gamma = (\vec{h}, 2\vec{e}_1) E_\gamma,$$

Weyl generators are labeled by vectors α called roots. All possible roots provides the root system of $sl(4)$.

$$\Delta^+_{C_2} \equiv \{\vec{e}_1 - \vec{e}_2, \quad \vec{e}_1 + \vec{e}_2, \quad 2\vec{e}_1, \quad 2\vec{e}_2, \}$$

and negative roots:

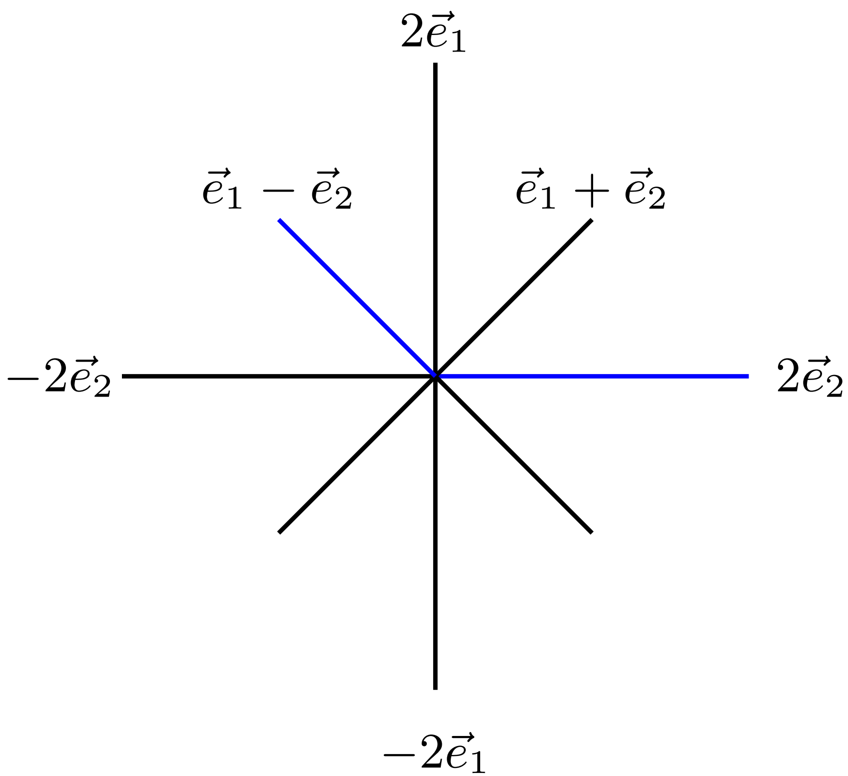
$$\Delta^-_{C_2} \equiv \{-\vec{e}_1 + \vec{e}_2, \quad -\vec{e}_1 - \vec{e}_2, \quad -2\vec{e}_1, \quad -2\vec{e}_2\}.$$

E_α with $\alpha > 0$ is upper-triangular; E_β with $\beta < 0$ is lower-triangular;

$$E_{-\alpha} = E_\alpha^T.$$

Long roots $\pm 2\vec{e}_1$ and $\pm 2\vec{e}_2$ with length 4; short roots $\pm \vec{e}_1 \pm \vec{e}_2$ with length 2.
Cartan-Weyl basis for $sp(4)$ in the 4-dimensional representation:

$$E_{e_1-e_2} = E_{12} + E_{34}, \quad E_{e_1+e_2} = E_{13} - E_{24}, \quad E_{2e_1} = E_{14}, \quad E_{2e_2} = E_{23},$$

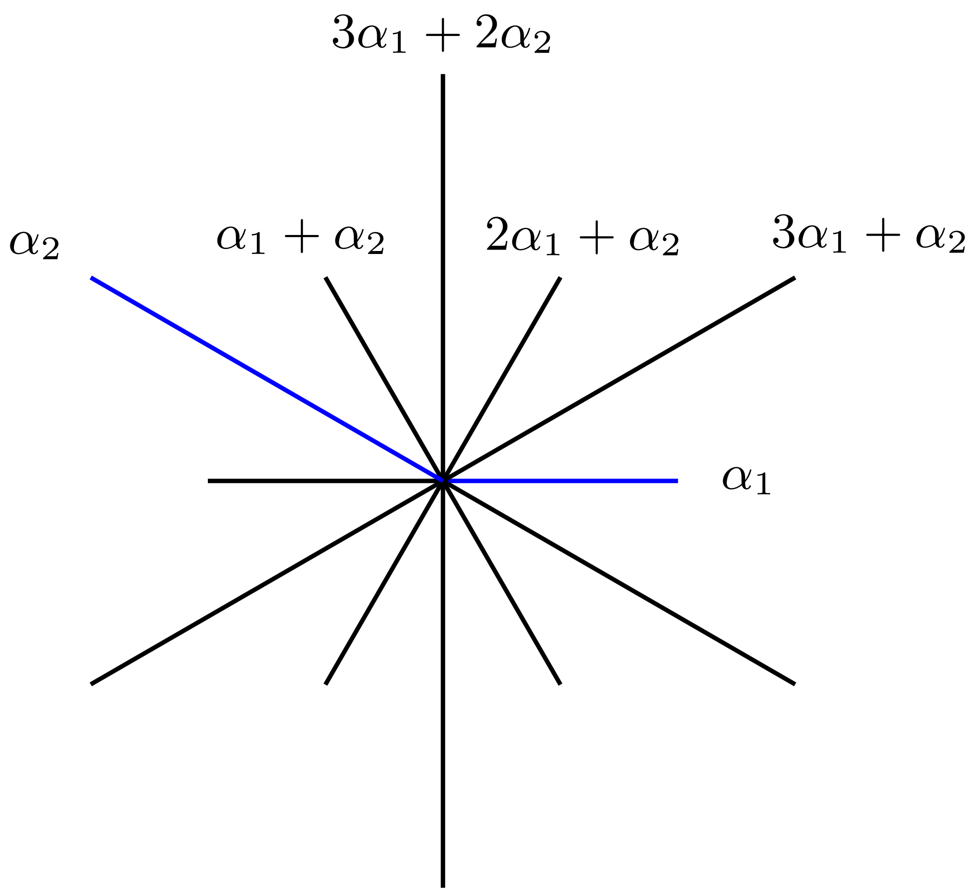


$C_3 \simeq sp(6)$, rank $sp(6) = 3$

$$\Delta^+_{B_3} \equiv \{\vec{e}_1 \pm \vec{e}_2, \quad \vec{e}_1 \pm \vec{e}_3, \quad \vec{e}_2 \pm \vec{e}_3, \quad 2\vec{e}_1, \quad 2\vec{e}_2, \quad 2\vec{e}_3, \}$$

Cartan-Weyl basis for $sp(6)$ in the 6-dimensional representation:

$$E_{e_j - e_k} = E_{jk} + (-1)^{j+k} E_{7-k,7-j}, \quad E_{e_j + e_k} = E_{j,7-k} + (-1)^{j+k} E_{k,7-j}, \quad E_{2e_k} = E_{k,7-k}.$$



The root system of the exceptional algebra $\mathfrak{g}_2$, $\alpha_1 = \vec{e}_1 - \vec{e}_2$, $\alpha_2 = -2\vec{e}_1 + \vec{e}_2 + \vec{e}_3$. All roots are orthogonal to $\vec{\epsilon} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$.

Cartan-Weyl basis of $D_4 \simeq so(8)$

Let $r = 4$. Then X is 8×8 matrix, such that

$$X + S_0 X^T S_0^{-1} = 0, \quad S_0 = \left(\begin{array}{c|c} 0_4 & s_0 \\ \hline \hat{s}_0 & 0_4 \end{array}\right), \quad s_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad S_0^2 = \mathbb{1}$$

Thus $\mathfrak{h}$ consists of diagonal matrices.

$$H = \left(\begin{array}{cccc|cccc} h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -h_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_1 \end{array}\right), \qquad \operatorname{tr} H = 0, \qquad \dim(\mathfrak{h}) = 4$$

Important: $H \in \mathfrak{h} \Leftrightarrow \vec{h} = (h_1, h_2, h_3, h_4)^T \in \mathbb{E}_4$. Then rank $so(8) = 4$.

Weyl generators E_α: eigenvalues of ad_H:

$$E_\alpha = \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right), \quad E_\beta = \left(\begin{array}{cccc|cccc} 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0\\ \hline 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{array}\right),$$

$$[H, E_\alpha] = (h_2 - h_3)E_\alpha = (\vec{h}, \vec{e}_2 - \vec{e}_3)E_\alpha, \qquad [H, E_\beta] = (h_2 + h_4)E_\beta = (\vec{h}, \vec{e}_2 + \vec{e}_4)E_\beta,$$

Weyl generators are labeled by vectors α called roots. All possible roots that provide the root system of $so(8)$ are:

$$\Delta^+_{D_4} \equiv \{\vec{e}_j - \vec{e}_k, \quad \vec{e}_j + \vec{e}_k, \quad 1 \leq j < k \leq 4\}$$

and negative roots:

$$\Delta^-_{D_4} \equiv \{-\vec{e}_j + \vec{e}_k, \quad -\vec{e}_j - \vec{e}_k, \quad 1 \leq j < k \leq 4\}.$$

E_α with $\alpha > 0$ is upper-triangular; E_β with $\beta < 0$ is lower-triangular;

$$E_{-\alpha} = E_\alpha^T.$$

All roots have the same length 2. The Weyl generators of D_4 in the 8-dimensional representation are given by:

$$E_{e_j - e_k} = E_{jk} - (-1)^{j+k} E_{9-k,9-j}, \qquad E_{e_j + e_k} = E_{j,9-k} - (-1)^{j+k} E_{k,9-j},$$

where

$$(E_{a,b})_{mn} = \delta_{am}\delta_{bn}.$$

Commutation relations and Killing form

The commutation relations between the Cartan–Weyl generators:

$$[H, E_\alpha] = (\alpha, \vec{h}) E_\alpha = \alpha(H) E_\alpha, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha,$$

where H_α is the Cartan element whose dual vector is α.

Besides

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta} & \text{if} \quad \alpha+\beta \in \Delta \\ 0 & \text{otherwize} \end{cases} .$$

The Killing form by definition is introduced in the adjoint representation:

$$B(X,Y) = \mathrm{tr}\,(\mathrm{ad}_X \mathrm{ad}_Y).$$

i.e. we first calculate X and Y in the adjoint representation: $X \to \mathrm{ad}_X$ and $Y \to \mathrm{ad}_Y$ and then calculate the trace.

Practically, **if we are using an irreducible representation of $\mathfrak{g}$,** we can assume that the Killing form, up to an overall constant is provided by:

$$B(X,Y) \simeq \mathrm{tr}\,(X,Y).$$

For the Cartan-Weyl basis we have:

$$B(H_\alpha, H_\beta) = (\alpha, \beta), \qquad B(H, E_\alpha) = 0, \qquad B(E_\alpha, E_{-\alpha}) = (\alpha, \alpha),$$

and if $\alpha + \beta \neq 0$ then

$$B(E_\alpha, E_\beta) = 0.$$

Important use of the Killing form. Let us assume that $X \in \mathfrak{g}$ is given by:

$$X = \sum_{j=1}^{r} x_j H_j + \sum_{\alpha \in \Delta} x_\alpha E_\alpha.$$

Given X we can recover its coefficients x_j and x_α using the Killing form as follows:

$$x_j = \frac{B(X, H_j^\vee)}{B(H_j, H_j^\vee)}, \qquad x_\alpha = \frac{B(X, E_{-\alpha})}{B(E_\alpha, E_{-\alpha})}.$$

where $H_j^\vee$ is a dual basis in $\mathfrak{h}$ such that

$$B(H_j, H_k^\vee) \simeq \delta_{jk}.$$

The Killing form is nondegenerate on $\mathfrak{h}$ and on $\mathfrak{g}$. This last property allows us to solve the direct and the inverse scattering problem for a given 'good' Lax operator.