INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME. 5

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- Lax representations, simple Lie algebras and spinor models
- Riemann-Hilbert problems and soliton solutions of spinor models

Spinor models: Lax representations and simple Lie algebras

Remember: Zakharov–Mikhailov derivation:

$$\Psi_x = U(x,t,\lambda)\Psi(x,t,\lambda), \qquad \Psi_t = V(x,t,\lambda)\Psi(x,t,\lambda),$$

$$U(x,t,\lambda) = \frac{U_1(x,t)}{\lambda - a}, \qquad V(x,t,\lambda) = \frac{V_1(x,t)}{\lambda + a},$$

where a is a real number. Choose

$$J = E_{1,N} - E_{N,1} = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \dots & 0 \end{pmatrix},$$

$$U_1 = \boldsymbol{\phi} J \boldsymbol{\phi}^{-1}(x,t), \qquad V_1 = \boldsymbol{\psi} J \boldsymbol{\psi}^{-1}(x,t),$$

Introduce the real vectors corresponding to the first and the last columns $\phi_{(1)}, \phi_{(N)}, \psi_{(1)}, \psi_{(N)}$:

$$\vec{a} = \begin{pmatrix} \phi_{1,1} \\ \vdots \\ \phi_{N,1} \end{pmatrix}, \quad \vec{c} = \begin{pmatrix} \phi_{1,N} \\ \vdots \\ \phi_{N,N} \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} \psi_{1,1} \\ \vdots \\ \psi_{N,1} \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} \psi_{1,N} \\ \vdots \\ \psi_{N,N} \end{pmatrix},$$

Then U_1 and V_1 acquire the form:

$$\text{if} \qquad Y = \vec{a}\vec{c}^T - \vec{c}\vec{a}^T \qquad \text{then} \qquad Y + Y^T = 0, \qquad \text{i.e.} \qquad Y \in so(N).$$

Then we introduce the complex vectors:

$$\vec{\phi} = \frac{1}{2}(\vec{a} + i\vec{c}), \qquad \vec{\psi} = \frac{1}{2}(\vec{u} + i\vec{v}).$$

i.e.

$$\phi_\alpha(x,t) = \frac{1}{2}(\phi^{(1)}_{\alpha,1} + i\phi^{(N)}_{\alpha,N}), \quad \psi_\alpha(x,t) = \frac{1}{2}(\psi^{(1)}_{\alpha,1} + i\psi^{(N)}_{\alpha,N})$$

Thus the ZM–system becomes:

$$i\frac{\partial\psi_\alpha}{\partial x} = \frac{1}{a}\sum_{\beta=1}^{N}(\phi_\alpha\phi_\beta^* - \phi_\alpha^*\phi_\beta)\psi_\beta, \qquad i\frac{\partial\phi_\alpha}{\partial t} = \frac{1}{a}\sum_{\beta=1}^{N}(\psi_\alpha\psi_\beta^* - \psi_\alpha^*\psi_\beta)\phi_\beta,$$

Alternative derivation:

$$\Psi_x = U(x,t,\lambda)\Psi(x,t,\lambda), \qquad \Psi_t = V(x,t,\lambda)\Psi(x,t,\lambda),$$

$$U(x,t,\lambda) = \frac{U_1(x,t)}{\lambda - a}, \qquad V(x,t,\lambda) = \frac{V_1(x,t)}{\lambda + a},$$

where a is a real number.

$$\frac{\partial U_1}{\partial t} + \frac{1}{2a}[U_1, V_1] = 0, \qquad \frac{\partial V_1}{\partial x} + \frac{1}{2a}[U_1, V_1] = 0,$$

Assume that $U_1, V_1 \in \mathfrak{g}$. Now we choose $\mathfrak{g} \simeq so(8)$ and introduce them by:

$$U_1(x,t) = \boldsymbol{\phi} H_{e_1} \boldsymbol{\phi}^{-1}(x,t), \qquad V_1(x,t) = \boldsymbol{\psi} H_{e_1} \boldsymbol{\psi}^{-1}(x,t),$$

$$\boldsymbol{\phi}(x,t) \in SO(8), \qquad \boldsymbol{\psi}(x,t) \in SO(8), \qquad H_{e_1} = \mathrm{diag}\,(1,0,0,0,0,0,0,-1).$$

$$\boldsymbol{\phi}^{-1}(x,t) = S_0 \boldsymbol{\phi}^T(x,t) S_0^{-1}, \qquad \boldsymbol{\psi}^{-1}(x,t) = S_0 \boldsymbol{\psi}^T(x,t) S_0^{-1}.$$

$$S_0 = \left(\begin{array}{c|c} 0_4 & s_0 \\ \hline \hat{s}_0 & 0_4 \end{array}\right), \quad s_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad S_0^2 = \mathbb{1}$$

Let us introduce:

$$\boldsymbol{\phi}(x,t) = (\vec{a}, \dots, \vec{c}), \qquad \boldsymbol{\phi}^{-1}(x,t) = \begin{pmatrix} \vec{c}^T S_0 \\ \vdots \\ \vec{a}^T S_0 \end{pmatrix},$$

$$\boldsymbol{\psi}(x,t) = (\vec{u}, \dots, \vec{v}), \qquad \boldsymbol{\phi}^{-1}(x,t) = \begin{pmatrix} \vec{v}^T S_0 \\ \vdots \\ \vec{u}^T S_0 \end{pmatrix},$$

Now U_1 and V_1 acquire the form:

$$U_1 = |\vec{a}\rangle\langle\vec{c}|S_0 - |\vec{c}\rangle\langle\vec{a}|S_0, \qquad V_1 = |\vec{u}\rangle\langle\vec{v}|S_0 - |\vec{v}\rangle\langle\vec{u}|S_0,$$

where the vectors $|\vec{a}\rangle$, $|\vec{c}\rangle$, $|\vec{u}\rangle$, $|\vec{v}\rangle$ are 8-component real vectors depending on x and t. Therefore:

$$U_1 + S_0 U_1^T S_0 = 0, \qquad V_1 + S_0 V_1^T S_0 = 0, \qquad \text{i.e.} \qquad U_1, V_1 \in so(8).$$

The equations for the vectors:

$$\frac{\partial|\vec{a}\rangle}{\partial t} - \frac{1}{2a} V_1 |\vec{a}\rangle = 0, \qquad \frac{\partial\langle\vec{a}|}{\partial t} + \langle\vec{a}|S_0 V_1 = 0,$$

$$\frac{\partial|\vec{c}\rangle}{\partial t} - \frac{1}{2a} V_1 |\vec{c}\rangle = 0, \qquad \frac{\partial\langle\vec{c}|}{\partial t} + \langle\vec{c}|S_0 V_1 = 0,$$

and

$$\frac{\partial|\vec{u}\rangle}{\partial x} + \frac{1}{2a} U_1 |\vec{u}\rangle = 0, \qquad \frac{\partial\langle\vec{u}|}{\partial x} - \langle\vec{u}|S_0 U_1 = 0,$$

$$\frac{\partial|\vec{v}\rangle}{\partial x} + \frac{1}{2a} U_1 |\vec{v}\rangle = 0, \qquad \frac{\partial\langle\vec{v}|}{\partial x} - \langle\vec{v}|S_0 U_1 = 0,$$

Introduce:

$$|\vec{\phi}\rangle = \frac{1}{2}(|\vec{a}\rangle + i|\vec{c}\rangle), \qquad |\vec{\psi}\rangle = \frac{1}{2}(|\vec{u}\rangle + i|\vec{v}\rangle),$$

i.e.

$$|\vec{a}\rangle = |\vec{\phi}\rangle + |\vec{\phi}^{\,*}\rangle, \qquad i|\vec{c}\rangle = |\vec{\phi}\rangle - |\vec{\phi}^{\,*}\rangle,$$
$$|\vec{u}\rangle = |\vec{\psi}\rangle + |\vec{\psi}^{\,*}\rangle, \qquad i|\vec{v}\rangle = |\vec{\psi}\rangle - |\vec{\psi}^{\,*}\rangle,$$

Therefore:

$$i(|\vec{a}\rangle\langle\vec{c}| - |\vec{c}\rangle\langle\vec{a}|) = -2(|\vec{\phi}\rangle\langle\vec{\phi}^{\,*}| - |\vec{\phi}^{\,*}\rangle\langle\vec{\phi}|),$$
$$i(|\vec{u}\rangle\langle\vec{v}| - |\vec{v}\rangle\langle\vec{u}|) = -2(|\vec{\psi}\rangle\langle\vec{\psi}^{\,*}| - |\vec{\psi}^{\,*}\rangle\langle\vec{\psi}|),$$

Now the ZM–system becomes:

$$i\frac{\partial|\vec{\psi}\rangle}{\partial x} = \frac{1}{a}\left(|\vec{\phi}\rangle\langle\vec{\phi}^{\,*}|S_0|\vec{\psi}\rangle - |\vec{\phi}^{\,*}\rangle\langle\vec{\phi}|S_0|\vec{\psi}\rangle\right),$$
$$i\frac{\partial|\vec{\phi}\rangle}{\partial t} = \frac{1}{a}\left(|\vec{\psi}^{\,*}\rangle\langle\vec{\psi}|S_0|\vec{\phi}\rangle - |\vec{\psi}\rangle\langle\vec{\psi}^{\,*}|S_0|\vec{\phi}\rangle\right),$$

i.e. we get the same model but with simple change of variables.

Riemann-Hilbert problem:

Given the sewing function $\Psi_0(x,\lambda)$ for $\lambda \in \mathbb{R}$ construct the fundamental analytic solutions $\xi^+(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_+$ and $\xi^-(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_-$ such that they satisfy the canonical normalization and eq. (**??**).

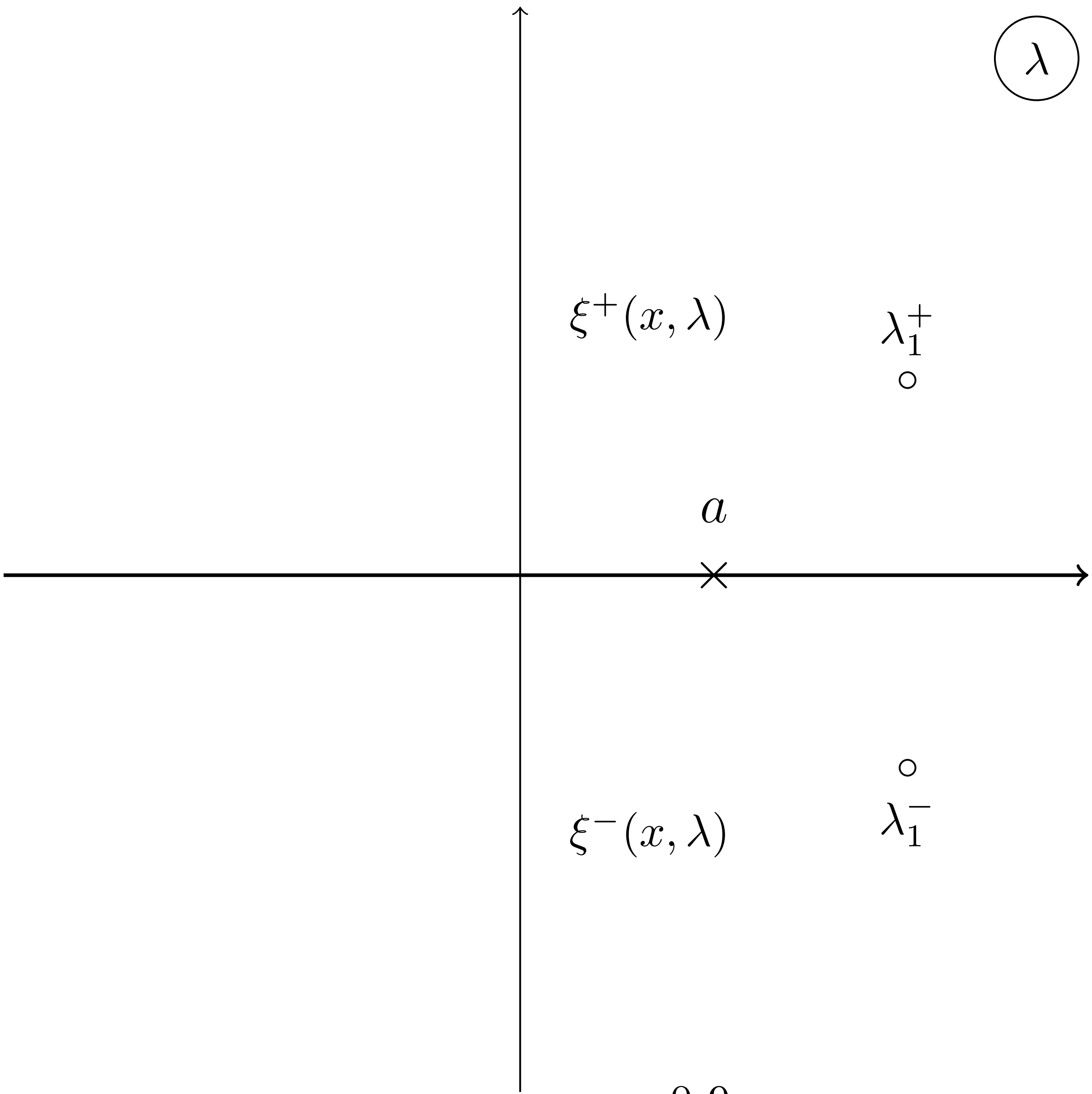
Remember: $\xi^\pm(x,\lambda)$ **satisfy the equations:**

$$\begin{aligned} i\frac{\partial \xi^\pm}{\partial x} &= \frac{U_1(x)}{\lambda - a}\xi^\pm(x,t,\lambda) - \xi^\pm(x,t,\lambda)\frac{J}{\lambda - a}, \\ i\frac{\partial \xi^\pm}{\partial t} &= \frac{V_1(x)}{\lambda + a}\xi^\pm(x,t,\lambda) - \xi^\pm(x,t,\lambda)\frac{J}{\lambda + a}. \end{aligned} \tag{1}$$

If we find the solution of the Riemann-Hilbert problem, the we can immediately find also $U_1(x)$. Multiply eq. (1) by $\lambda - a$ and by $\hat{\xi}^\pm(x,\lambda)$ on the right, then take the limit $\lambda \to a$:

$$\lim_{\lambda\to a} \quad : \quad i(\lambda - a)\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,\lambda) = U_1(x) - \xi^\pm J\hat{\xi}^\pm(x,\lambda),$$

$$
\begin{aligned}
U_1(x) &= \lim_{\lambda\to a}\left(\xi^{\pm} J\hat{\xi}^{\pm}(x,\lambda) - i(\lambda-a)\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right) \\
&= J - i\lim_{\lambda\to a}(\lambda-a)\left(\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right).
\end{aligned}
\tag{2}
$$



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RHP:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \qquad \xi^\pm, G \in so(8).$$

$$i\frac{\partial G}{\partial x} = \frac{1}{\lambda - a}[J, G(x,t,\lambda)], \qquad i\frac{\partial G}{\partial t} = \frac{1}{\lambda + a}[J, G(x,t,\lambda)],$$

$$G(x,t,\lambda) = \mathcal{E}_0^{-1} G_0(\lambda)\mathcal{E}_0(x,t,\lambda), \qquad \mathcal{E}_0(x,t,\lambda) = \exp\left(-\frac{iJx}{\lambda - a} - \frac{iJt}{\lambda + a} \right).$$

Let $\xi_0(x,t,\lambda)$ be a regular solution to RHP. Construct singular solution of RHP with simple pole singularities at λ_1^+ and λ_1^-.

First construct singular solutions with canonical normalization at $\lambda = \infty$.

$$\xi_1(x,t,\lambda) = u(x,t,\lambda)\xi_0(x,t,\lambda),$$

$$u(x,t,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)\, P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \bar{P}_1(x,t), \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}.$$

$$P_1(x,t) = \frac{|n_1\rangle\langle m_1|}{\langle m_1|n_1\rangle}, \qquad \bar{P}_1(x,t) = S_0 P_1^T(x,t) S_0 = \frac{|S_0 m_1\rangle\langle n_1 S_0|}{\langle m_1|n_1\rangle},$$

The projectors P_1 and $\bar{P}_1(x,t)$ and the polarization vectors must satisfy:

$$P_1 \bar{P}_1(x,t) = \bar{P}_1(x,t) P_1 = 0, \qquad \langle m_1 | S_0 | m_1 \rangle = \langle n_1 | S_0 | n_1 \rangle = 0.$$

Then

$$S_0 u^T(x,t,\lambda) S_0 = u^{-1}(x,t,\lambda).$$

However, we need normalization at $\lambda = a$. The corresponding solutions are:

$$\tilde{\xi}^+(x,t,\lambda) = \xi^+(x,t,\lambda)\hat{\xi}^+(x,t,a), \qquad \tilde{\xi}^-(x,t,\lambda) = \xi^-(x,t,\lambda)\hat{\xi}^-(x,t,a).$$

$$\tilde{u}(x,t,\lambda) = u(x,t,\lambda)\hat{u}(x,t,a),$$

$$\begin{aligned} \tilde{u}(x,t,\lambda) &= \left(\mathbb{1} + (c_1(\lambda) - 1) P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \bar{P}_1(x,t) \right) \\ &\quad \cdot \left(\mathbb{1} + (c_1(a) - 1) \bar{P}_1(x,t) + \left(\frac{1}{c_1(a)} - 1 \right) P_1(x,t) \right) \\ &= \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(a)} - 1 \right) P_1(x,t) + \left(\frac{c_1(a)}{c_1(\lambda)} - 1 \right) \bar{P}_1(x,t). \end{aligned} \tag{3}$$