INTEGRABLE SPINOR MODELS IN TWO DIMENSIONAL SPACE-TIME. 6

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- Riemann-Hilbert problems and soliton solutions of Zakharov–Mikhailov models

The Lax representation for the ZM system

Start with the ZM–system:

$$
\begin{aligned}
i\frac{\partial|\vec{\psi}\rangle}{\partial x} &= \frac{1}{a}\left(|\vec{\phi}\rangle\langle\vec{\phi}^{\,*}|S_0|\vec{\psi}\rangle - |\vec{\phi}^{\,*}\rangle\langle\vec{\phi}|S_0|\vec{\psi}\rangle\right),\\
i\frac{\partial|\vec{\phi}\rangle}{\partial t} &= \frac{1}{a}\left(|\vec{\psi}^{\,*}\rangle\langle\vec{\psi}|S_0|\vec{\phi}\rangle - |\vec{\psi}\rangle\langle\vec{\psi}^{\,*}|S_0|\vec{\phi}\rangle\right),
\end{aligned}
$$

It allows Lax representation:

$$
\Psi_x = U(x,t,\lambda)\Psi(x,t,\lambda), \qquad \Psi_t = V(x,t,\lambda)\Psi(x,t,\lambda),
$$

$$
U(x,t,\lambda) = \frac{U_1(x,t)}{\lambda - a}, \qquad V(x,t,\lambda) = \frac{V_1(x,t)}{\lambda + a},
$$

$$
U_1(x,t) = \boldsymbol{\phi} H_{e_1} \boldsymbol{\phi}^{-1}(x,t), \qquad V_1(x,t) = \boldsymbol{\psi} H_{e_1} \boldsymbol{\psi}^{-1}(x,t),
$$

$$
\boldsymbol{\phi}(x,t) \in SO(8), \qquad \boldsymbol{\psi}(x,t) \in SO(8), \qquad H_{e_1} = \mathrm{diag}\,(1,0,0,0,0,0,0,-1).
$$

The scattering problem for L

$$L: \quad i\Psi_x = \frac{U_0(x,t)}{\lambda - a}\Psi(x,t,\lambda), \qquad M: \quad i\Psi_t = \frac{V_0(x,t)}{\lambda + a}\Psi(x,t,\lambda),$$

$$U_0 = \phi J \phi^\dagger, \qquad V_0 = \psi J \psi^\dagger.$$

In fact there is indeterminacy in the second operator:

$$M: \quad i\Psi_t = \frac{V_0(x,t)}{\lambda + a}\Psi(x,t,\lambda) - \Psi(x,t,\lambda)C(\lambda),$$

where $C(\lambda)$ will be determined below.

Boundary conditions, i.e. the limits of U_0 and V_0 for $x \to \pm\infty$. For the spinor models the natural boundary conditions are

$$\lim_{x\to\pm\infty} \psi(x,t) = \mathbb{1}_N, \quad \lim_{x\to\pm\infty} \phi(x,t) = \mathbb{1}_N, \quad \lim_{x\to\pm\infty} U_0(x) = J, \quad \lim_{x\to\pm\infty} V_0(x) = J,$$

Asymptotic solutions:

$$i\Psi_{0,x} = \frac{J}{\lambda - a}\Psi_0(x,\lambda), \qquad \Psi_0(x,\lambda) = \exp\left(\frac{-iJx}{\lambda - a}\right).$$

Jost solutions:

$$\lim_{x\to\infty} \Psi(x,t,\lambda)\Psi_0^{-1}(x,\lambda) = \mathbb{1}, \qquad \lim_{x\to-\infty} \Phi(x,t)\Psi_0^{-1}(x,\lambda) = \mathbb{1}.$$

and the scattering matrix. Due to the special choice of J and to the fact that the Jost solutions and the scattering matrix take values in the group $SO(N)$ we can use the following block-matrix structure of $T(\lambda,t)$

$$T(\lambda,t) = \psi^{-1}\phi(x,t,\lambda) = \left(\begin{array}{ccc} m_1^- & \vec{b}^{-T} & c_1^- \\ -\vec{B}^+ & \mathbf{T}_{22} & s_0\vec{B}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{array}\right), \qquad (1)$$

where $\vec{b}^{\pm}(\lambda,t)$ and $\vec{B}^{\pm}(\lambda,t)$ are $N-2$-component vectors, $\mathbf{T}_{22}(\lambda)$ is a $(N-2)\times(N-2)$ block and $m_1^{\pm}(\lambda)$, $c_1^{\pm}(\lambda)$ are scalar functions.

Jost solutions – integral equations

$$\Psi(x,\lambda) = \Psi_0(x,\lambda) - \frac{i}{\lambda - a}\int_{\infty}^{x} dy\, \Psi_0(x-y,\lambda)(U_0(y) - J)\Psi(y,\lambda),$$

$$\Phi(x,\lambda) = \Psi_0(x,\lambda) - \frac{i}{\lambda - a}\int_{-\infty}^{x} dy\, \Psi_0(x-y,\lambda)(U_0(y) - J)\Phi(y,\lambda),$$

$$\Psi_0(x-y,\lambda) = \exp\left(-\frac{i(x-y)}{\lambda - a}(1,0,0,\dots,0,-1)\right)$$

$$\Psi_0(x-y,\lambda) = \exp\left(\mathrm{Im}\left(\frac{(x-y)}{\lambda - a}\right)(1,0,0,\dots,0,-1) + \mathit{oscillating}\right)$$

Assume, that $\lim_{x\to\pm\infty} U_1(x) = J$**. If** $\mathrm{Im}\,(\lambda - a) > 0$**, then** $\mathrm{Im}\,\frac{1}{\lambda - a} < 0$**. Then:**

- If $\mathrm{Im}\,(\lambda - a) = 0$, then $\Psi_0(x-y,\lambda)$ oscillates and $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ are well defined!

- Consider $\Psi(x,\lambda)$: we have $y > x$ and therefore $\Psi_{0;11}(x-y,\lambda)$ decreases for $\mathrm{Im}\,\frac{1}{\lambda - a} > 0$; $\Psi_{0;NN}(x-y,\lambda)$ decrease for $\mathrm{Im}\,\frac{1}{\lambda - a} < 0$; the rest matrix elements $\Psi_{0;kk}(x-y,\lambda)$ oscillate. Therefore analytic extensions are possible for the first and the last columns only:

$$\Psi(x,\lambda) = \left(\Psi^{+}_{(1)}(x,\lambda), \vec{\Psi}(x,\lambda), \Psi^{-}_{(N)}(x,\lambda)\right)$$

- Consider $\Phi(x,\lambda)$: we have $y<x$ and now the situation is opposite:

$$\Phi(x,\lambda) = \left(\Phi^-_{(1)}(x,\lambda), \vec{\Phi}(x,\lambda), \Phi^+_{(N)}(x,\lambda)\right)$$

Fundamental analytic solutions

Important tools for reducing the ISP to a Riemann-Hilbert problem (RHP) are the fundamental analytic solution (FAS) $\chi^{\pm}(x,t,\lambda)$. Their construction is based on the generalized Gauss decomposition of $T(\lambda,t)$:

$$T(t,\lambda) = S_J^+ D_J^+ \hat{T}_J^-, \qquad \text{or} \qquad T(t,\lambda) = S_J^- D_J^- \hat{T}_J^+,$$

$$\chi^+(x,t,\lambda) = \phi(x,t,\lambda) T_J^-(t,\lambda) = \psi(x,t,\lambda) S_J^+(t,\lambda) D_J^+(\lambda),$$
$$\chi^-(x,t,\lambda) = \phi(x,t,\lambda) T_J^+(\lambda) = \psi(x,t,\lambda) S_J^-(t,\lambda) D_J^-(\lambda),$$

where

$$T_J^+(\lambda,t) = \begin{pmatrix} 1 & -\vec{\rho}^{+,T} & \tilde{c}^+ \\ 0 & \mathbb{1} & -s_0\vec{\rho}^+ \\ 0 & 0 & 1 \end{pmatrix}, \quad T_J^-(\lambda,t) = \begin{pmatrix} 1 & 0 & 0 \\ \vec{\rho}^- & \mathbb{1} & 0 \\ \tilde{c}^- & -\vec{\rho}^{-,T} s_0 & 1 \end{pmatrix},$$

$$\vec{\rho}^{+} = \frac{\vec{b}^{+}}{m_1^{+}}, \quad \vec{\rho}^{-} = \frac{\vec{B}^{-}}{m_1^{-}}, \quad \tilde{c}^{+} = \frac{1}{2}(\vec{\rho}^{+,T} s_0 \vec{\rho}^{+}), \quad \tilde{c}^{-} = \frac{1}{2}(\vec{\rho}^{-,T} s_0 \vec{\rho}^{-}).$$

$$S_J^{+}(\lambda, t) = \begin{pmatrix} 1 & \vec{\tau}^{+,T} & \tilde{c}^{+} \\ 0 & \mathbb{1} & s_0 \vec{\tau}^{+} \\ 0 & 0 & 1 \end{pmatrix}, \quad S_J^{-}(\lambda, t) = \begin{pmatrix} 1 & 0 & 0 \\ -\vec{\tau}^{-} & \mathbb{1} & 0 \\ \tilde{c}^{-} & -\vec{\tau}^{-,T} s_0 & 1 \end{pmatrix},$$

$$\vec{\tau}^{+} = \frac{\vec{B}^{-}}{m_1^{+}}, \quad \vec{\tau}^{-} = \frac{\vec{b}^{+}}{m_1^{-}}, \quad \tilde{c}^{+} = \frac{1}{2}(\vec{\tau}^{+,T} s_0 \vec{\tau}^{+}), \quad \tilde{c}^{-} = \frac{1}{2}(\vec{\tau}^{-,T} s_0 \vec{\tau}^{-}).$$

$$D_J^{+} = \begin{pmatrix} m_1^{+} & 0 & 0 \\ 0 & \mathbf{m}_2^{+} & 0 \\ 0 & 0 & 1/m_1^{+} \end{pmatrix}, \qquad D_J^{-} = \begin{pmatrix} 1/m_1^{-} & 0 & 0 \\ 0 & \mathbf{m}_2^{-} & 0 \\ 0 & 0 & m_1^{-} \end{pmatrix}, \tag{2}$$

where $\vec{\tau}^{\pm}(\lambda, t) = \vec{b}^{\mp}/m_1^{\pm}$, $\vec{\rho}^{\pm}(\lambda, t) = \vec{b}^{\pm}/m_1^{\pm}$ and

$$\mathbf{m}_2^{+} = \mathbf{T}_{22} + \frac{\vec{b}^{+}\vec{b}^{-T}}{m_1^{+}}, \qquad \mathbf{m}_2^{-} = \mathbf{T}_{22} + \frac{s_0\vec{b}^{-}\vec{b}^{+T}s_0}{m_1^{-}}.$$

Scattering matrix and its t-dependence

$$\Phi(x,t,\lambda) = \Psi(x,t,\lambda)T(t,\lambda), \quad T(\lambda,t) = \begin{pmatrix} m_1^- & \vec{b}^{-T} & c_1^- \\ -\vec{B}^+ & \mathbf{T}_{22} & s_0\vec{B}^- \\ c_1^+ & -\vec{b}^{+T}s_0 & m_1^+ \end{pmatrix},$$

Consider the limit for $x \to -\infty$:

$$i\frac{\partial\Phi}{\partial t} = \frac{V_1}{\lambda+a}\Phi(x,t,\lambda) - \Phi(x,t,\lambda)C(\lambda) \qquad x \to -\infty,$$

$$\Rightarrow \qquad i\frac{\partial\Psi_0}{\partial t} = \frac{J}{\lambda+a}\Psi_0(x,\lambda) - \Psi_0(x,\lambda)C(\lambda), \qquad C(\lambda) = \frac{J}{\lambda+a}.$$

because $\frac{\partial\Psi_0}{\partial t} = 0$! Next consider the limit for $x \to \infty$ with $\Phi = \Psi T$:

$$\Rightarrow \qquad i\Psi_0(x,\lambda)\frac{\partial T}{\partial t} = \frac{J}{\lambda+a}\Psi_0(x,\lambda)T(t,\lambda) - \Psi_0(x,\lambda)T(t,\lambda)C(\lambda).$$

Finally:

$$i\frac{\partial T}{\partial t} = \left[\frac{J}{\lambda+a}, T(t,\lambda)\right].$$

In components we get:

$$\frac{\partial m_1^{\pm}(\lambda)}{\partial t} = 0, \qquad \frac{\partial \boldsymbol{m}_2^{\pm}(\lambda)}{\partial t} = 0, \quad i\frac{\partial \vec{b}^{\pm}}{\partial t} = \mp \frac{2\vec{b}^{\pm}(\lambda, t)}{\lambda + a}, \quad i\frac{\partial \vec{B}^{\pm}}{\partial t} = \mp \frac{2\vec{B}^{\pm}(\lambda, t)}{\lambda + a},$$

Thus $m_1^-(\lambda)$ and $m_1^+(\lambda)$: i) are analytic functions of λ for $\operatorname{Im}\lambda < 0$ and $\operatorname{Im}\lambda > 0$; ii) provide generating functionals of conservation laws for the spinor models. Usually for other models we use:

$$\ln m_1^-(\lambda) = \sum_{k=1}^{\infty} I_k \lambda^{-k}; \qquad \frac{\partial I_k}{\partial t} = 0.$$

and I_k come out to have densities, which are local in the dynamical variables; besides I_k are in involution, i.e. the Poisson brackets $\{I_k, I_m\} = 0$ In this case we need to check if I_k will be local or nonlocal in $\vec{\phi}$ and $\vec{\psi}$.

Besides we have a whole $(N-2)\times(N-2)$ matrix $\boldsymbol{m}_2^+(\lambda)$ that also generates integrals of motion. Each matrix element of $\boldsymbol{m}_2^+(\lambda)$ generates conservation laws, but in general we can not expect neither local densities, nor vanishing Poisson brackets between these integrals.

The FAS for real λ are linearly related

$$\chi^+(x,t,\lambda) = \chi^-(x,t,\lambda) G_J(\lambda,t), \qquad G_{0,J}(\lambda,t) = \hat{T}_J^-(\lambda,t) T_J^+(\lambda,t). \qquad (3)$$

Introduce:

$$\xi^\pm(x,t,\lambda) = \chi^\pm(x,t,\lambda)\hat{\Psi}_0(x,\lambda).$$

Then the RHP can be written as:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda) G_J(x,t,\lambda), \qquad \lambda \in \mathbb{R},$$

$$i\frac{\partial G_J}{\partial x} = \frac{1}{\lambda - a}[J, G_J(x,t,\lambda)], \qquad i\frac{\partial G_J}{\partial t} = \frac{1}{\lambda + a}[J, G_J(x,t,\lambda)].$$

$$\lim_{\lambda \to a} \xi^\pm(x,t,\lambda) = \mathbb{1}. \qquad (4)$$

Obviously the sewing function $G_j(x,\lambda,t)$ is uniquely determined by the Gauss factors $T_J^\pm(\lambda,t)$ and

$$G(x,t,\lambda) = \mathcal{E}_0^{-1} G_0(\lambda) \mathcal{E}_0(x,t,\lambda), \qquad \mathcal{E}_0(x,t,\lambda) = \exp\left(-\frac{iJx}{\lambda - a} - \frac{iJt}{\lambda + a} \right).$$

Let $\xi_0(x,t,\lambda)$ be a regular solution to RHP. Construct singular solution of RHP with simple pole singularities at λ_1^+ and λ_1^-.

First construct singular solutions with canonical normalization at $\lambda = \infty$.

$$\xi_1(x,t,\lambda) = u(x,t,\lambda)\xi_0(x,t,\lambda) \qquad \text{and} \qquad \chi_1^\pm(x,t,\lambda) = u(x,t,\lambda)\chi_0^\pm(x,t,\lambda),$$

$$u(x,t,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)\, P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1 \right) \bar{P}_1(x,t), \qquad c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}.$$

$$P_1(x,t) = \frac{|n_1\rangle\langle m_1|}{\langle m_1|n_1\rangle}, \qquad \bar{P}_1(x,t) = S_0 P_1^T(x,t) S_0 = \frac{|S_0 m_1\rangle\langle n_1 S_0|}{\langle m_1|n_1\rangle},$$

Since $\chi_1^\pm(x,t,\lambda)$ and $\chi_0^\pm(x,t,\lambda)$ satisfy the equations:

$$i\frac{\partial \chi_1^\pm}{\partial x} = \frac{U_1(x,t)}{\lambda - a}\chi_1^\pm(x,t,\lambda), \qquad \frac{\partial \chi_0^\pm(x,t,\lambda)}{\partial x} = \frac{J}{\lambda - a}\chi_0^\pm(x,t,\lambda),$$

Then the dressing factor must satisfy the equation:

$$i\frac{\partial u}{\partial x} = \frac{U_1(x)}{\lambda - a} u(x,t,\lambda) - u(x,t,\lambda)\frac{J}{\lambda - a}. \qquad (5)$$

identically with respect to λ. But $u(x,t,\lambda)$ is rational function of λ which has poles and zeroes at $\lambda=\lambda_1^+$ and $\lambda=\lambda_1^-$. This means that it is enough to require that the residues of the left hand side of (5) at these points vanish. From this it follows:

$$|n_1(x,t)\rangle = \mathcal{E}(x,t,\lambda_1^+)|n_{10}\rangle, \qquad \langle m_1(x,t)| = \langle m_{10}|\mathcal{E}^{-1}(x,t,\lambda_1^-). \qquad (6)$$

In other words the dependence of the projector $P_1(x,t)$ is determined by the eigenfunctions of L with trivial potential $U_0 = J$, since $\chi_0^{\pm}(x,t,\lambda) = \mathcal{E}(x,t,\lambda)$.

Besides $u(x,t,\lambda)$ must be an element of the $SO(N)$ group, i.e.

$$u^{-1}(x,t,\lambda) = S_0 u^T(x,t,\lambda) S_0,$$

which means that the projectors P_1 and $\bar{P}_1(x,t)$ and the polarization vectors must satisfy:

$$P_1\bar{P}_1(x,t) = \bar{P}_1(x,t)P_1 = 0, \qquad \langle m_1|S_0|m_1\rangle = \langle n_1|S_0|n_1\rangle = 0.$$

If we need normalization at $\lambda = a$, the we could use:

$$\tilde{\xi}^+(x,t,\lambda) = \xi^+(x,t,\lambda)\hat{\xi}^+(x,t,a), \qquad \tilde{\xi}^-(x,t,\lambda) = \xi^-(x,t,\lambda)\hat{\xi}^-(x,t,a).$$

$$\tilde{u}(x,t,\lambda) = u(x,t,\lambda)\hat{u}(x,t,a),$$

$$\begin{aligned}\tilde{u}(x,t,\lambda) &= \left(\mathbb{1} + (c_1(\lambda)-1)\,P_1(x,t) + \left(\frac{1}{c_1(\lambda)}-1\right)\bar{P}_1(x,t)\right) \\ &\quad\cdot\left(\mathbb{1} + (c_1(a)-1)\,\bar{P}_1(x,t) + \left(\frac{1}{c_1(a)}-1\right)P_1(x,t)\right) \\ &= \mathbb{1} + \left(\frac{c_1(\lambda)}{c_1(a)}-1\right)P_1(x,t) + \left(\frac{c_1(a)}{c_1(\lambda)}-1\right)\bar{P}_1(x,t).\end{aligned} \tag{7}$$

The one soliton solution

Given the sewing function $\Psi_0(x,\lambda)$ for $\lambda \in \mathbb{R}$ construct the fundamental analytic solutions $\xi^+(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_+$ and $\xi^-(x,\lambda)$ analytic for $\lambda \in \mathbb{C}_-$ such that they satisfy the canonical normalization $\lim_{\lambda\to\infty} \xi^-(x,\lambda) = \mathbb{1}$.

Remember: $\xi^\pm(x,\lambda)$ **satisfy the equations:**

$$\begin{aligned} i\frac{\partial \xi^\pm}{\partial x} &= \frac{U_1(x)}{\lambda - a}\xi^\pm(x,t,\lambda) - \xi^\pm(x,t,\lambda)\frac{J}{\lambda - a}, \\ i\frac{\partial \xi^\pm}{\partial t} &= \frac{V_1(x)}{\lambda + a}\xi^\pm(x,t,\lambda) - \xi^\pm(x,t,\lambda)\frac{J}{\lambda + a}. \end{aligned} \tag{8}$$

If we find the solution of the Riemann-Hilbert problem, the we can immediately find also $U_1(x)$. Multiply eq. (8) by $\lambda - a$ and by $\hat{\xi}^\pm(x,\lambda)$ on the right, then take the limit $\lambda \to a$:

$$\lim_{\lambda\to a} \quad : \quad i(\lambda - a)\frac{\partial \xi^\pm}{\partial x}\hat{\xi}^\pm(x,\lambda) = U_1(x) - \xi^\pm J \hat{\xi}^\pm(x,\lambda),$$

i.e.

$$U_1(x,t) = \lim_{\lambda\to a}\left(\xi^{\pm} J\hat{\xi}^{\pm}(x,\lambda) - i(\lambda - a)\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right). \tag{9}$$

Since we started with $\xi^{\pm}(x,t,\lambda) = u(x,t,\lambda)\xi_0^{\pm}$ and $\xi_0^{\pm} = \mathbb{1}$, then

$$U_1(x,t) = u(x,t,a)J\hat{u}(x,t,a).$$

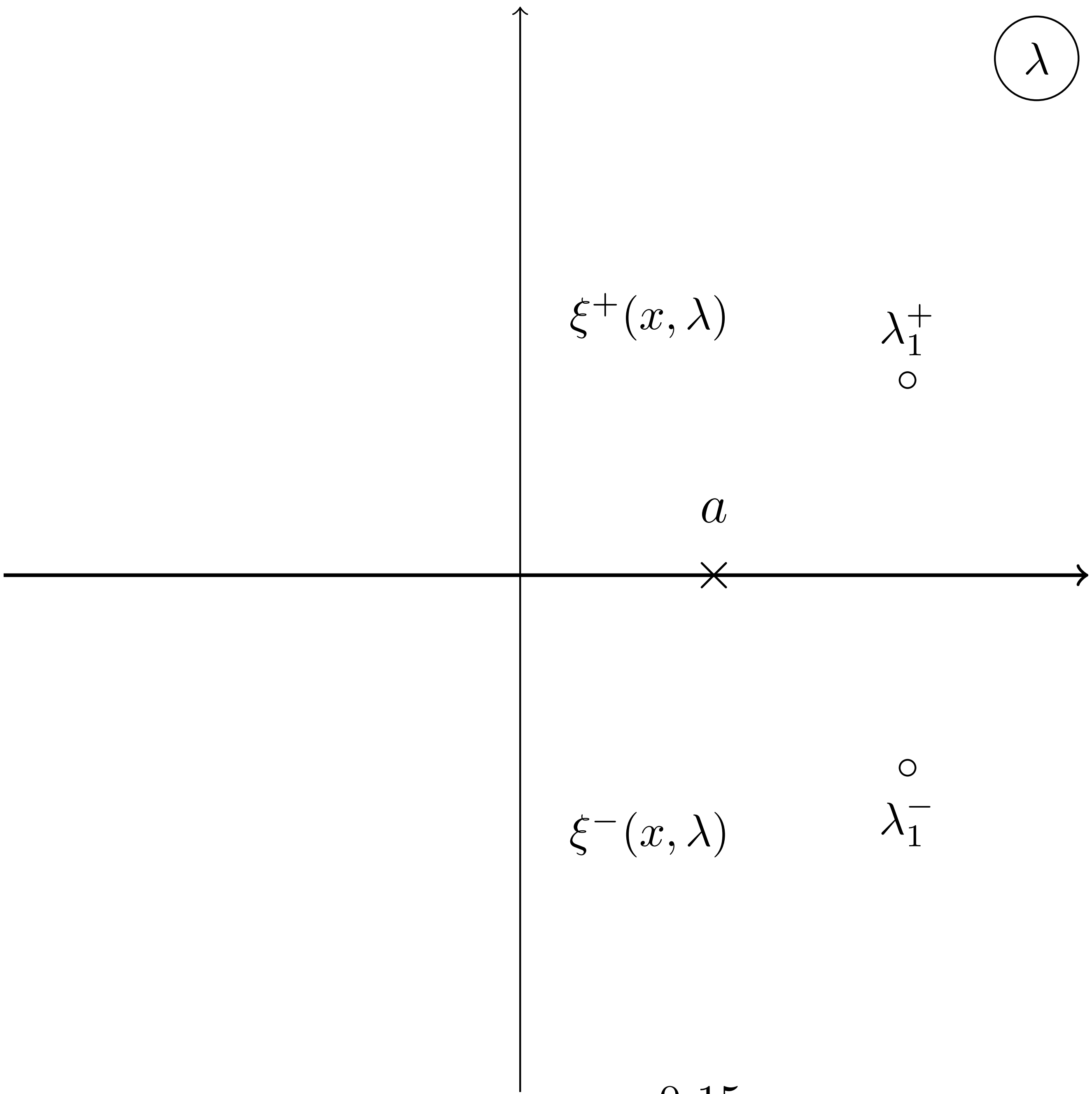
where $u(x,t,\lambda)$ is the dressing factor, which we constructed above. Similarly:

$$\begin{aligned} V_1(x,t) &= \lim_{\lambda\to -a}\left(\xi^{\pm} J\hat{\xi}^{\pm}(x,\lambda) - i(\lambda + a)\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,\lambda)\right) \\ &= u(x,t,-a)J\hat{u}(x,t,-a) \end{aligned} \tag{10}$$

How to calculate $\vec{\phi}(x,t)$ and $\vec{\psi}(x,t)$? Assume we start with the trivial solution, corresponding to $\xi_0^{\pm} = \mathbb{1}$. Then $\vec{\phi}_0 =$ const , $\vec{\psi}_0 =$ const with $(\vec{\phi}_0^*|S_0|\vec{\psi}_0) = 0$ and $(\vec{\phi}_0|S_0|\vec{\psi}_0) = 0$. Then the dressed solutions will be:

$$\vec{\phi}(x,t) = u(x,t,a)\vec{\phi}_0, \qquad \vec{\psi}(x,t) = u(x,t,-a)\vec{\psi}_0.$$

Check it


λ
ξ⁺(x, λ)
λ₁⁺
a
ξ⁻(x, λ)
λ₁⁻

RHP:

$$\xi^+(x,t,\lambda) = \xi^-(x,t,\lambda)G(x,t,\lambda), \qquad \lambda \in \mathbb{R}, \qquad \xi^\pm, G \in so(N).$$

$$i\frac{\partial G}{\partial x} = \frac{1}{\lambda - a}[J, G(x,t,\lambda)], \qquad i\frac{\partial G}{\partial t} = \frac{1}{\lambda + a}[J, G(x,t,\lambda)],$$

Several versions of getting N-soliton solutions via dressing method:

- Start with $\xi_0^\pm = \mathbb{1}$; then construct $u_1(x,t,\lambda)$ and derive $\xi_1^\pm = u_1(x,t,\lambda)$. Now apply the dressing on $\xi_1^\pm$, derive $u_2(x,t,\lambda)$ and find $\xi_2^\pm = u_2(x,t,\lambda)u_1(x,t,\lambda)$. etc, etc.

- Use projectors of higher rank, e.g.:

$$P(x,t) = \sum_{k,j=1}^{2} |n_k\rangle \widehat{M}_{kj} \langle m_j|, \qquad M_{kj} = \langle m_j|n_k\rangle, \qquad \widehat{M} = M^{-1}.$$

  This provides one-soliton solution with more complicated internal structure.

- use more general dressing factors:

$$u(x,t,\lambda) = \mathbb{1} + \sum_{j=1}^{N} \left((c_j(\lambda) - 1) P_j + \left((c_j^{-1}(\lambda) - 1) \bar{P}_j \right) \right),$$

Impose constraints:

$$u(x,t,\lambda) S_0 u^T(x,t,\lambda) S_0 = \mathbb{1},$$

and

$$i\frac{\partial u}{\partial x} + \frac{U_1(x)}{\lambda - a} u(x,t,\lambda) - u(x,t,\lambda)\frac{J}{\lambda - a} = 0.$$

$$i\frac{\partial u}{\partial x} + \frac{V_1(x)}{\lambda + a} u(x,t,\lambda) - u(x,t,\lambda)\frac{J}{\lambda + a} = 0.$$

which must hold identically with respect to λ. This leads to a set of algebraic equations on $P_j(x,t)$.